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COMPLEX NUMBER

1.1 Introduction: We have discussed about Natural number, Real number, Rational number etc. Here we will introduce imaginary number. In the set of real number **R,** the equation $x^2 + 9 = 0$ has no solution.

Because here $x^2 = -9$ Hence $x = \pm\sqrt{-9}$ which is impossible.

Since we know that square of a (positive or negative) number is always positive. To get solution of the equation, mathematician **Euler** introduced a symbol iota denoted by i, which is the first letter of the word imaginary. For the square root of -1, he introduce the property

$-1 = i^2$ $[i^2 = i.i = \sqrt{-1}\sqrt{-1} = (\sqrt{-1})^2 = -1]$

Now $\sqrt{-9}$ can be written as $\sqrt{-9} = \sqrt{-1}\sqrt{9} = i3$

1.2 Positive power of *i*

$i^4 = i^2.i^2 = (-1)(-1) = 1;$ $i^{18} = (i^2)^9 = (-1)^9 = -1$

1.3 Definition of Complex Number :

If a and b are two real numbers then a number of the form a + i b where $i = \sqrt{-1}$ is called a **Complex Number.** eg. $3 + 5i$, $4 - 7i$, $-7 + 4i$ etc.

Note: 1. The Complex Number a + ib can be written as a + bi.

2. Let $z = a + i\, b$ then 'a' is called real part of z and we write it as $\mathrm{Re}(z) = a$ Similarly 'b' is called imaginary part of z and we write it as $\mathrm{Im}(z) = b$

3. Every real number can be expressed as $r = r + i0$ (ie.$a + i\, b$ form). So it is also a complex Number.

1.4 Equality of Complex Number : Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two Complex Numbers. Then $z_1 = z_2$ if $a_1 = a_2$ and $b_1 = b_2$

1.5 Operation on Complex Number.

(i) Addition of Complex Number : Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two Complex Numbers. Now, $z_1 + z_2 = (a_1 + i\, b_1) + (a_2 + i\, b_2) = (a_1 + a_2) + i(b_1 + b_2)$

$= \mathrm{M} + i\mathrm{N}$ where $\mathrm{M} = (a_1 + a_2)$ and $\mathrm{N} = b_1 + b_2$

Addition of Complex Numbers is also a Complex Number.

(ii) Subtraction of Complex Number: Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two Complex Numbers. Now, $z_1 - z_2 = (a_1 + i\, b_1) - (a_2 + i\, b_2) = (a_1 - a_2) + i\,(b_1 - b_2)$

$= \mathrm{M} + i\mathrm{N}$ where $\mathrm{M} = (a_1 - a_2)$ and $\mathrm{N} = (b_1 - b_2)$

Subtraction of two or more Complex Numbers is also a Complex Number.

(iii) Multiplication of Complex Number : Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two Complex Numbers. Now, $z_1 \,.\, z_2 = (a_1 + i\, b_1) \,.\, (a_2 + i\, b_2)$

$= a_1\, a_2 + i\, a_2 b_1 + i\, a_1 b_2 + i^2 b_1 b_2$

$= (a_1 a_2 - b_1 b_2) + i\,(a_2 b_1 + a_1 b_2) = \mathrm{M} + i\mathrm{N}$ where $\mathrm{M} = (a_1 a_2 - b_1 b_2)$ and $\mathrm{N} = (a_2 b_1 + a_1\, b_2)$

Multiplication of two or more Complex Numbers is also a Complex Number**.**

(iv) Division of Complex Number : Let $z_1 = a_1 + i\, b_1$ and $z_2 = a_2 + i\, b_2$ be two Complex Numbers. Now, $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$

$$= \frac{a_1 a_2 + i(a_2 b_1 - a_1 b_2) - i^2 b_1 b_2}{(a_2)^2 - (ib_2)^2} =$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2{}^2 + b_2{}^2} + i\,\frac{a_2 b_1 - a_1 b_2}{a_2{}^2 + b_2{}^2} = \mathrm{M} + i\mathrm{N}$$

where $M = \frac{a_1a_2+b_1b_2}{a_2{}^2+b_2{}^2}$ and $N = \frac{a_2b_1-a_1b_2}{a_2{}^2+b_2{}^2}$

Division of two or more Complex Numbers is also a Complex Number.

1.6 Conjugate of Complex Number :

If $z = a + i\,b$ is a complex number then its conjugate complex number will be $a - i\,b$. Conjugate complex number is denoted by $\bar{z}$. Or $\bar{z} = a - i\,b$.

1.7 Properties of Conjugate of Complex Number:

(i) Addition of a complex number and its conjugate complex number is a real number.

Thus $z + \bar{z} = (a + i\,b) + (a - i\,b) = 2a$, which is a real number.

(ii) Subtraction of a complex number and its conjugate complex number is an imaginery number.

Thus $z - \bar{z} = (a + i\,b) - (a - i\,b) = i2b$ which is an imaginary number.

(iii) $z = \bar{z} \Rightarrow z$ is purely real.

(iv) $z.\bar{z} = (a + i\,b)(a - i\,b) = a^2 + b^2$

1.8 Modulus of Complex Number :

Modulus of the complex number $a + ib = \sqrt{a^2+b^2}$. i.e if $z = a + ib$, where a and b are real, $i = \sqrt{-1}$, $a^2 + b^2 \neq 0$, then $\sqrt{a^2+b^2}$ (that is positive square root of $a^2 + b^2$) is known as **Modulus** of $a + ib$ and is denoted by $|z|$ or mod(z).

Thus **mod(z) = mod ($a + ib$) = $+\sqrt{a^2+b^2}$** or $|z| = |a + ib| = +\sqrt{a^2+b^2}$

1.9 Amplitude of the complex number : The amplitude of the complex number $x + iy$ may be written as $\tan^{-1}\frac{y}{x}$

If $z = x + iy$ then **amp z = $\tan^{-1}\frac{y}{x}$** or, **arg z = $\tan^{-1}\frac{y}{x}$**

Note: If $x = r\cos\theta$ and $y = r\sin\theta$ then $x + iy = r\cos\theta + i\,r\sin\theta$
$= r(\cos\theta + i\sin\theta)$

Thus any complex number can be expressed as $\boldsymbol{r(\cos\theta + i\sin\theta)}$ which is the **polar** form.

1.10 Properties of modulus

(i) Modulus of a complex number and its conjugate complex no is same.
Let $z = a + ib$ then $\bar{z} = a - ib$
$|z| = |a + ib| = \sqrt{a^2+b^2} = \sqrt{a^2+(-b)^2} = |\bar{z}|$

(ii) Modulus of the product of two complex numbers is equal to the product of modulus of those complex numbers.

(iii) Modulus of the quotient of two complex number is equal to the quotient of modulus of those comp numbers.

(iv) $|z\bar{z}| = |z||\bar{z}| = |z|^2 = |\bar{z}|^2$

1.11 Properties of amplitude :

(i) amp $z = -$ amp $\bar{z}$

(ii) For z_1 and z_2 two complex numbers amp $(z_1.z_2) = \text{amp}(z_1) + \text{amp}(z_2)$

(iii) For z_1 and z_2 two complex numbers amp $\frac{z_1}{z_2} = \text{amp}(z_1) - \text{amp}(z_2)$

1.12 Square root of a complex number :

1.13 Cube root of unity :

If ω is the imaginary cube root of unity then the other imaginary cube root of unity is ω^2.

$\therefore$ 1, ω, ω^2 are the cube roots of unity.

Then $\omega^2 + \omega + 1 = 0$ and Also $\omega^3 = 1$.

Worked out examples.

Ex 1. Reduce $\frac{1+i}{1-i}$ to $a + ib$ form

Soln : $z = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = i = 0 + i$ Re(z) = 0, Im(z) = 1

Ex 2. I) [QP 2011] $\sqrt{-81} + \sqrt{-64}$

Soln: $\sqrt{-81} + \sqrt{-64} = 9i + 8i = 17i$

ii) [QP 2011] $i^2 + \frac{i}{i^2}$

Soln: $i^2 + \frac{1}{i^2} = -1 - 1 = -2$

(iii) [QP 2010] Transform $(2 + i)(2 - 3i)(4 - 3i)$ into A+ iB form.

Soln: $(2 + i)(2 - 3i)(4 - 3i) = (4 + 2i - 6i - 3i^2)(4 - 3i) = (4 - 4i + 3)(4 - 3i)$
$= (7 - 4i)(4 - 3i) = 28 - 16i - 21i + 12\,i^2$
$= 28 - 37i - 12 = 16 - 37i = 16 + (-37)i$
$= A + i\,B$
where A =16 and B = –37

Ex 3.[QP 2009, 2013, 2015] Prove that $\sqrt{i} + \sqrt{-i} = \sqrt{2}$

Proof: LHS $= \sqrt{i} + \sqrt{-i} = \sqrt{(\sqrt{i} + \sqrt{-i})^2} = \sqrt{i + (-i) + 2.i(-i)}$
$= \sqrt{2}$ = RHS Proved.

Ex 4.(i) Find out the Modulus of each of the following:

(a) $3 - 4i$ **(b)[Q.P 2010]** $\frac{3+4i}{12+5i}$

Solution: (a) mod$(3 - 4i) = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

(b) $\left|\frac{3+4i}{12+5i}\right| = \frac{|3+4i|}{|12+5i|} = \frac{\sqrt{3^2+4^2}}{\sqrt{12^2+5^2}} = \frac{5}{13}$

(ii) Find the argument of the complex number

Ex.5 (a)[Q.P 2009] $\sqrt{3} + i$ **(b)** $\frac{1+2i}{1-3i}$

Solution: (a) $\arg(\sqrt{3} + i) = tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

(b) $\arg\frac{1+2i}{1-3i} = \arg\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)} = \arg\frac{1-6+5i}{1+9} = \arg\frac{-5+5i}{1+9}$
$= \arg(-\frac{1}{2} + i\frac{1}{2}) = tan^{-1}\frac{1/2}{-1/2} = \frac{3\pi}{4}$

(the complex number is plotted in 2nd quadrant of the argand plane.)

Ex6. (i) [QP 2011] If $x = 1 + i$ find the value of $x^2 - 2x + 2$

Soln: $x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x-1)^2 + 1 = (1 + i - 1)^2 + 1 = i^2 + 1 = 0$

(ii) If $x = 3 + 2i$ and $y = 3 - 2i$ find the value of $x^2 + xy + y^2$

Soln: $x^2 + xy + y^2 = (x^2 + y^2) + xy = (3+2i)^2 + (3-2i)^2 + (3+2i)(3-2i)$

$= 2\{3^2 + (2i)^2\} + (9 + 4) = 10 + 13 = 23$

Ex 7. Find the square root: $3 + 4i$

Soln: Let $\sqrt{3+4i} = x + iy \Rightarrow x^2 - y^2 = 3$, $2xy = 4$, $x^2 + y^2 = \sqrt{3^2 + 4^2} = 5$

$\Rightarrow x^2 = (3+5)/2 = 4 \Rightarrow x = \pm 2$

$y^2 = (5-3)/2 = 1 \Rightarrow y = \pm 1$

since $xy > 0$ so x and y are of same sign. Hence $\sqrt{3+4i} = \pm(2 + i)$

Ex 8. If is imaginary cube root of unity prove that:

(i)[QP 2009, 2013] $(1 - \omega + \omega^2)^2 + (1 + \omega - \omega^2)^2 = -4$

(ii)[QP 2012] $(1 - \omega + \omega^2)^3 + (1 + \omega - \omega^2)^3 = -16$

(iii)(hw)[QP 2011, 2015] $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

Solution:

(i) $(1 - \omega + \omega^2)^2 + (1 + \omega - \omega^2)^2 = (1 + \omega - 2\omega + \omega^2)^2 + (1 + \omega + \omega^2 - 2\omega^2)^2$

$= (0 - 2\omega)^2 + (0 - 2\omega^2)^2 = 4(\omega^2 + \omega^4) = 4(\omega^2 + \omega) = -4$

(ii) $(1 - \omega + \omega^2)^3 + (1 + \omega - \omega^2)^3 = (1 + \omega - 2\omega + \omega^2)^3 + (1 + \omega + \omega^2 - 2\omega^2)^3$

$= (0 - 2\omega)^3 + (0 - 2\omega^2)^3 = -8(\omega^3 + \omega^6) = -8(1 + 1) = -16$

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2. Variation

1.1 Direct variation : when two quantities are so related that if one of them is changed , the other is changed in the same ratio .

In other words we can say ,if x_1, x_2 be any two values of a quantity X and y_1, y_2 be the corresponding values of a second quantity Y then X is said to be vary directly as Y when $x_1 : x_2 = y_1 : y_2$

Note : The symbol ∞ is used to denote the word varies as .

So , $a \infty b$ is read as a varies as b .

$a \infty b$, then the ratio a : b is constant .

If K is the constant of variation then a : b = K $\Rightarrow \frac{a}{b} = K \quad \therefore a = Kb$

1.2 Indirect variation or inverse variation : One quantity X is said to be vary inversely as another quantity Y when X varies directly as the reciprocal of Y .

Thus if P varies inversely as Q i.e $P \infty \frac{1}{Q}$ then $P = K\frac{1}{Q}$ where K is constant .

Note: If $P \propto \frac{1}{Q}$ then $P = K\frac{1}{Q}$ where K is constant .

$\Rightarrow PQ = K$, i.e constant

1.3 Joint variation: When $X \propto YZ$, X is said to vary jointly with Y and Z .

1.4 Some elementary result.

(i) If $p \propto q$, then $q \propto p$.

(ii) If $p \propto q$, then $p^n \propto q^n$

(iii) If $p \propto q$, then $q \propto r$ then $p \propto r$.

(iv) If $p \propto r$, then $q \propto r$ then $p \pm q \propto r$.

(v) If $p \propto r$, then $q \propto r$ then $\sqrt{pq} \propto r$.

(vi) If $p \propto qr$, then $q \propto \frac{p}{r}$ and $r \propto \frac{p}{q}$

(vii) If $p \propto q$, then $r \propto s$ then $pr \propto qs$.

(viii) If $p \propto q$, then $r \propto s$ then $\frac{p}{r} \propto \frac{q}{s}$.

Worked out examples :

Ex.1 $x \propto y$ and if $x= 3$, when $y = 9$, when Find x , when $y = 3$.

Soln : According to question , $x \propto y$ ie. $x = Ky$ ……..(i)

Given , when $y = 9, x = 3$ $\therefore$ $3 = K.9$ K is constant

$$\Rightarrow K = \frac{3}{9} = \frac{1}{3}$$

Put $K = \frac{1}{3}$ in (i) $x = \frac{1}{3}y$

Now if $y = 3$ then $x = \frac{1}{3}.3 = 1$

$\therefore$ $x = 1$ Ans.

Ex.2 x varies directly as y^2 and $y = 4$ when $x=8$ then find y when $x = 32$.

Soln: x varies directly as y^2 $\Rightarrow x \propto y^2$

$\Rightarrow x = Ky^2$ ……………….(i)

Given $y = 4$ when $x=8$

$$\therefore\ 8 = K4^2 \Rightarrow \frac{8}{16} = K \Rightarrow \frac{1}{2} = K$$

Put K in (i) $x = \frac{1}{2}y^2$

when $x = 32$, $32 = \frac{1}{2}y^2$ $64 = y^2 \Rightarrow y = \pm 8$ Ans .

Ex.3 If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$ then $z \propto x$

Proof : Given , $x \propto \frac{1}{y}$ $\Rightarrow$ $x = M\frac{1}{y}$ $\Rightarrow xy = M$

Again $y \propto \frac{1}{z} \Rightarrow y = N\frac{1}{z} \Rightarrow yz = N$

Now , $\frac{xy}{yz} = \frac{M}{N} \Rightarrow \frac{x}{z} = L$ $\quad$ ($\frac{M}{N}$ = L is a constant)

$\Rightarrow x = Lz \Rightarrow z = \frac{1}{L}x \Rightarrow z \propto x.$ Proved .

Ex.4 If $x + y \propto x - y$ then prove that $x^3 + y^3 \propto x^3 - y^3$

Proof : Given $x + y \propto x - y$

$\Rightarrow x + y = M(x - y)$
$\Rightarrow x + y = Mx - My$
$\Rightarrow y + My = Mx - x$
$\Rightarrow y(1+M) = (M-1)x$
$\Rightarrow y(1+M) = (M-1)x$
$\Rightarrow y = \frac{M-1}{M+1}x$

$\Rightarrow y = Lx$ $\quad$ ($\frac{M-1}{M+1}$ = L which is constant)

$\Rightarrow y^3 = L^3 x^3$ $\quad$ (cubing both side)
$\Rightarrow y^3 = P x^3$ ……………(i) $\quad$ ($L^3 = P$ which is constant)

Now , $\frac{x^3+y^3}{x^3-y^3} = \frac{x^3+Px^3}{x^3-Px^3} = \frac{x^3(1+P)}{x^3(1-P)} = \frac{(1+P)}{(1-P)} = R$ $\quad$ { $\frac{(1+P)}{(1-P)}$ = R which is a constant }

$\therefore \frac{x^3+y^3}{x^3-y^3} = R$

$\Rightarrow x^3 + y^3 = R(x^3 - y^3) \Rightarrow x^3 + y^3 \propto x^3 - y^3$ Proved .

Ex .5 If $x + y \propto x - y$ then prove that $x^2 + y^2 \propto xy$

Proof : Given $x + y \propto x - y$

$\Rightarrow x + y = M(x - y)$
$\Rightarrow x + y = Mx - My \Rightarrow y + My = Mx - x \Rightarrow y(1+M) = (M-1)x$
$\Rightarrow y(1+M) = (M-1)x$
$\Rightarrow y = \frac{M-1}{M+1}x \Rightarrow y = Lx$

($\frac{M-1}{M+1}$ = L which is constant)

Now , $\frac{x^2+y^2}{xy} = \frac{x^2+L^2x^2}{xLx} = \frac{x^2(1+L^2)}{Lx^2} = \frac{(1+L^2)}{L} = P$ $\quad$ $\frac{(1+L^2)}{L} = P$, which is constant .

$\therefore \frac{x^2+y^2}{xy} = P \Rightarrow x^2 + y^2 = Pxy$

$\Rightarrow x^2 + y^2 \propto xy$

Ex6. If $x^2 + y^2 \propto x^2 - y^2$ then prove that $x \propto y$

Proof : Given $x^2 + y^2 \propto x^2 - y^2$

$\Rightarrow x^2 + y^2 = M(x^2 - y^2)$

$$\Rightarrow x^2 + y^2 = M x^2 - My^2$$
$$\Rightarrow y^2 + My^2 = Mx^2 - x^2$$
$$\Rightarrow y^2 (1+ M) = (M - 1) x^2$$
$$\Rightarrow y^2 \frac{1+M}{M-1} = x^2$$
$$\Rightarrow x^2 = Ly^2 \qquad \left(\frac{1+M}{M-1} = L \text{ which is constant .}\right)$$
$$\Rightarrow x = \sqrt{L}\, y \qquad (\sqrt{L} \text{ is constant})$$
$$\Rightarrow x \propto y . \quad \text{Proved .}$$

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3. Logarithm

There are two system of Logarithm

1. Common Logarithm
2. Napier Logarithm (or Natural Logarithm)

Common Logarithm : It was first introduced by Prof. Henry Briggs. It is a kind of Logarithm developed with respect to the base '10' of decimal system. It is used for numeric computation .

Napier Logarithm: (or Natural Logarithm): It was first introduced by Jhon Napiar(1550-1617) It is a kind of Logarithm developed with respect to the base 'e' where $2<e<3$ ($e=2.7183$). It is used for theoretical purpose .

1.1 Definition of Logarithm : Let us consider the equation $a^x = N$ ($a>0, a \neq 1$) ,where a is called base and x is the index of the power .
Now x is called Logarithm of N to the base a and is written as $x = \log_a N$. This is read as Logarithm of N to the base a .

Examples: (i) If $3^2 = 9$ then $2 = \log_3 9$ (ii) If $3^{(-2)} = \frac{1}{9}$ then $-2 = \log_3 \frac{1}{9}$

1.2 Laws of Logarithm :

Law 1. $\log_a(m \times n) = \log_a m + \log_a n$, ($m ,n >0$)

Similarly $\log_a(m \times n \times p) = \log_a m + \log_a n + \log_a p$

Law 2. $\log_a(\frac{m}{n}) = \log_a m - \log_a n$, ($m ,n >0$)

Law 3. $\log_a(m)^n = n\log_a m$

1.3 Special cases in logarithm :

Case 1. Since $a^0 = 1$ therefore $\log_a 1= 0$ .Thus $\log_5 1= 0$, $\log_9 1= 0$
Case 2. Since $a^1 = a$ therefore $\log_a a = 1$.Thus $\log_5 5= 1$, $\log_9 9= 1$
Case 3. Since $a^{-1} = \frac{1}{a}$ therefore $\log_a \frac{1}{a} = -1$.Thus $\log_5 \frac{1}{5} = -1$, $\log_9 \frac{1}{9} = -1$

1.4 Properties of logarithm: (change of base)

$\log_a m = \log_b m \times \log_a b$

cor 1. $\log_a b \times \log_b a =1$

Proof : Put m = a in $\log_a m = \log_b m \times \log_a b$ we have

$$\log_a a = \log_b a \times \log_a b$$

$$\therefore \quad 1 = \log_b a \times \log_a b$$

cor 2. $\log_a m = \log_b m / \log_b a$

cor 3. $\log_b a \times \log_c b \times \log_a c = 1$

Proof : $\log_b a \times \log_c b \times \log_a c = \log_c a \times \log_a c = 1$

Worked out example:

Ex1. Find the logarithm of 125 to the base $\sqrt{5}$

Sol^n : We know if $a^x = N$ then $x = \log_a N$

Here N = 125 , a = $\sqrt{5}$

Therefore $\sqrt{5}^x = 125$

$\sqrt{5}^x = (\sqrt{5})^6$

$\therefore x = 6$ $\therefore$ Required logarithm is 6

Ex 2. Find the base of the logarithm of 1728 is 6 .

Soln. We know if $a^x = N$ then $x = \log_a N$

Here N = 1728 , x = 6

Therefore $(a)^6 = 1728$

$(a)^6 = (2\sqrt{3})^6$

$\therefore a = 6$ $\therefore$ Required base of the logarithm is 6 .

Ex 3. Find the value of $\log_6 216$

Sol^n $\log_6 216 = \log_6 6^3$

$= 3 \log_6 6 = 3.1 = 3$. $(\log_a a = 1)$

Ex.4 Prove that $\log_2\log_{\sqrt{2}}\log_3 81 = 2$

Sol^n : $\log_2\log_{\sqrt{2}}\log_3 81$

$= \log_2\log_{\sqrt{2}}\log_3 3^4$

$= \log_2\log_{\sqrt{2}} 4\log_3 3$

$= \log_2\log_{\sqrt{2}} 4$ $\quad$ $(\log_3 3 = 1)$

$= \log_2\log_{\sqrt{2}} \sqrt{2}^4$

$= \log_2 4\log_{\sqrt{2}} \sqrt{2}$

$= \log_2 4$ $\quad$ $(\log_{\sqrt{2}} \sqrt{2} = 1)$

$= \log_2 2^2$

$= 2 \log_2 2$

$= 2$ $\quad$ $(\log_2 2 = 1)$

Ex.5 If $a^2+b^2 = 7ab$ then Prove that $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

Sol^n : Given $a^2+b^2 = 7ab$

$\Rightarrow a^2+b^2 + 2ab = 9ab$

$\Rightarrow (a+b)^2 = 9ab$

$$\Rightarrow \frac{(a+b)^2}{9} = ab \Rightarrow \frac{(a+b)}{3} = \sqrt{ab} \Rightarrow \frac{(a+b)}{3} = (ab)^{1/2} \Rightarrow \log\frac{(a+b)}{3} = \log(ab)^{½}$$

$$\Rightarrow \log\frac{(a+b)}{3} = \frac{1}{2}\log(ab)$$

$$\Rightarrow \log\frac{(a+b)}{3} = \frac{1}{2}(\log a + \log b) \quad \text{proved .}$$

Ex. 6 If $\log\frac{a+b}{3} = \frac{1}{2}$ (loga+logb) then Prove that $\frac{a}{b} + \frac{b}{a} = 7$

Sol^n : (same as above problem only in reverse order)

Then we will get $a^2+b^2 = 7ab$

divied both side by ab, we have $\frac{a}{b} + \frac{b}{a} = 7$

Ex.7 If $\log\frac{1}{4}(x+2y) = \frac{1}{2}$ (logx+logy) then Prove that $x^2+4y^2 = 12xy$.

Sol^n : Given $\log\frac{1}{4}(x+2y) = \frac{1}{2}$ (logx+logy)

$$\Rightarrow \log\frac{1}{4}(x+2y) = (\log x + \log y)^{1/2}$$

$$\Rightarrow \log\frac{1}{4}(x+2y) = (\log x.y)^{1/2}$$

$$\Rightarrow \log\frac{1}{4}(x+2y) = \log \sqrt{x.y}$$

$$\Rightarrow \{\frac{1}{4}(x+2y)\} = (\sqrt{xy})$$

$$\Rightarrow \{\frac{1}{4}(x+2y)\}^2 = (\sqrt{xy})^2 \quad \text{(Squaring both side)}$$

$$\Rightarrow \frac{1}{16}(x^2+4y^2+4xy) = \log xy$$

$$\Rightarrow (x^2+4y^2+4xy) = 16\log xy \Rightarrow x^2+4y^2 = 12\log xy \quad \text{proved .}$$

Ex .8 Prove that $x^{\log y - \log z}.y^{\log z - \log x}.z^{\log x - \log y} = 1$

Sol^n: Let $x^{\log y-\log z}.\ y^{\log z-\log x}\ .\ z^{\log x-\log y} = u$

$\therefore$ $\text{Log}(x^{\log y-\log z}.\ y^{\log z-\log x}\ .\ z^{\log x-\log y}) = \log u$

$\Rightarrow$ $\text{Log}x^{\log y-\log z} + \log y^{\log z-\log x} + \log z^{\log x-\log y} = \log u$

$\Rightarrow$ $\text{Log}x^{\log y-\log z} + \log y^{\log z-\log x} + \log z^{\log x-\log y} = \log u$

$\Rightarrow$ (logy-logz)logx + (logz-logx)log y + (logx-logy) log z = log u

$\Rightarrow$logylogx - logz logx + logzlogy - logx log y + logxlogz - logy log z = log u

$\Rightarrow$ 0 = log u

$\Rightarrow$ Log 1= log u

$\Rightarrow$u=1

$\therefore$ $x^{\log y-\log z}.\ y^{\log z-\log x}\ .\ z^{\log x-\log y} = 1$. proved

Ex.9 Prove that $23\log\frac{16}{15} + 17\log\frac{25}{24} + \log\frac{81}{80} = 1$

Sol^n : L.H.S = $23\log\frac{16}{15} + 17\log\frac{25}{24} + \log\frac{81}{80}$

$=23\log\frac{16}{15} + 17\log\frac{25}{24} + 10\log\frac{81}{80}$
= 23(log16-log15) + 17(log25 – log24) + 10(log81 –log 80)
=23(log16-log15) + 17(log25 – log24) + 10(log81 – log80)
= $23(\log 2^4 - \log 3.5) + 17(\log 5^2 - \log 3.2^3) + 10(\log 3^4 - \log 5.2^4)$
= $23\{4\log 2 - (\log 3 + \log 5)\} + 17\{2\log 5 - (\log 3 + \log 2^3)\} + 10\{4\log 3 - (\log 5 + \log 2^4)\}$
= 92log2-23(log3+log5)+34log5 – 17 (log3+3log2) + 40log3 – 10(log 5+4log2)
= 92log2-23log3 -23log5+ 34log5 – 17log3-51log2+ 40log3 – 10 log 5- 40log2
= 52log2 - 51log2- 40log3+ 40log3+ 34log5 - 33log5
= log2 + log5
= log10 =1. Proved

Ex .10 Solve $\log_x 2 + \log_x 4 + \log_x 64 = 9$

Sol^n: $\log_x 2 + \log_x 4 + \log_x 64 = 9$
$\Rightarrow \log_x (2.4.64) = 9$
$\Rightarrow \log_x (512) = 9$
$\Rightarrow x^9 = 512$
$\Rightarrow x^9 = 2^9 \Rightarrow x = 2$

Ex .11 Solve $\log_x(8x-3) - \log_x 4 = 2$
$\Rightarrow \log_x \frac{8x-3}{4} = 2$
$\Rightarrow x^2 = \frac{8x-3}{4}$
$\Rightarrow 4x^2 - 8x + 3 = 0$
$\Rightarrow 4x^2 - 6x - 2x + 3 = 0$
$\Rightarrow 2x(2x-3) - 1(2x-3) = 0 \Rightarrow (2x-1).(2x-3) = 0$ Either $(2x-1)=0$ or $(2x-3) = 0$
$\Rightarrow x = \frac{1}{2}$ $\Rightarrow x = \frac{3}{2}$

****

4
Quadratic Equation

1.1 An equation in which the highest power of unknown quantity is two is called a quadratic equation . Quadratic equation is generally expressed in the form $ax^2+bx+c=0$, $a \neq 0$

1.2 A quadratic equation can not have more than two roots .

1.3 Nature of the roots :
Let the roots of quadratic equation $ax^2 + bx + c = 0$, have two roots α and β .

$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$\sqrt{b^2-4ac}$ is called **Discriminant** of the quadratic equation . The nature of the roots depend on this discriminant

1) if $b^2-4ac > 0$ then $\sqrt{b^2-4ac}$ is real . $\therefore$ the roots are real and unequal

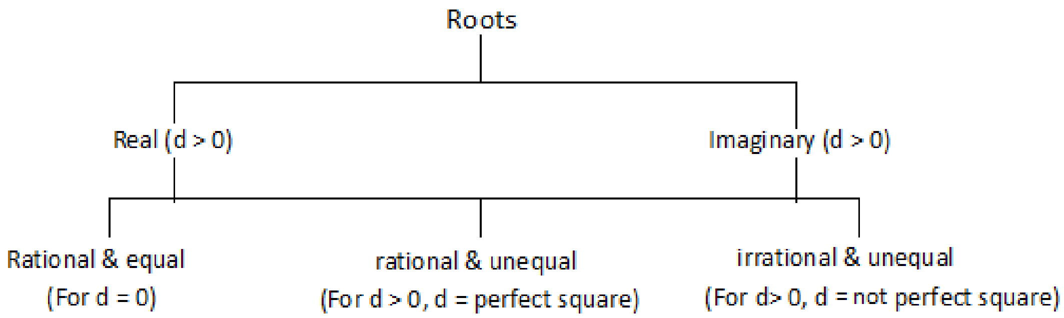
2) if $b^2-4ac=0$ then roots are real and equal and each equal to $\frac{-b}{a}$.

3) $b^2-4ac < 0$ then $\sqrt{b^2-4ac}$ is imaginary . $\therefore$ the roots are imaginary and unequal

4) if b^2-4ac is positive and perfect square then the roots are rational and unequal . provided a,b,c are rational .

but if b^2-4ac is positive and not a perfect square then the roots are irrational and unequal .

TABULAR FORM(of nature of roots)



1.4 Relation between roots and co-efficient

We have $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Sum of roots = $\alpha + \beta = \dfrac{-b}{a} = \dfrac{-coeff.of x}{co.eff.of x^2}$; product of roots = $\alpha \times \beta = \dfrac{c}{a} = \dfrac{const}{co.eff.of x^2}$

1.5 NOTE: Roots in special cases: In $ax^2+bx+c=0$,

1) If c = 0 ie. The constant term vanish then $\alpha \times \beta = \dfrac{c}{a} = \dfrac{0}{a} = 0$

$\therefore \alpha \times \beta = 0$ ie one of the two roots is zero

.

2) If b= 0 ie. The coeff. of x vanish then $\alpha + \beta = \dfrac{-b}{a} = \dfrac{-0}{a} = 0$

$\therefore \alpha = -\beta$

ie the roots are equal in magnitude but opposite in direction .

3) If b= 0 and c= 0 ie. If coeff. of x vanish and constant term vanish then equation become $ax^2 = 0 \Rightarrow x^2 = 0$ ($a \neq 0$) have both the roots are zero .

5) If a= c ie. The coeff. of x^2 = the constant term then $\alpha \times \beta = \dfrac{c}{a} = \dfrac{0}{a} = 1$, Then $\alpha = \dfrac{1}{\beta}$

ie the one root is reciprocal to the other .

1.6 Formation of quadratic equation from given roots .

Let α and β be the roots of $ax^2 + bx + c = 0$, $a \neq 0$

Then we have $\alpha + \beta = \dfrac{-b}{a}$ and $\alpha \times \beta = \dfrac{c}{a}$

Now $ax^2 + bx + c = 0$

$\Rightarrow x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$ (dividing by a)

$\Rightarrow x^2 -(-\frac{b}{a})x + \frac{c}{a} = 0$

$\Rightarrow x^2 -(\alpha+\beta)x + \alpha\,\beta = 0 \;\Rightarrow x^2 -($ Sum of roots$)x +$ product of roots $= 0$

1.7 Conjugate roots: (two theorem)

Theorem 1. In a quadratic equation with real coeff., irrational roots occur in pairs

Theorem 2. In a quadratic equation with real coeff., imaginary roots occur in pairs.

1.8 Condition for common root :

Case-1: When one root of two quadratic equation is common

Let α be a common root of quadratic equation $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$,

then $(a_2c_1 - a_1c_2)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$ which is the required condition .

Case-11: When both roots of two quadratic equation is common

Then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ which is the required condition .

Ex. 1 Discuss the nature of the roots .

(i) $2x^2 - 6x + 3 = 0$,

Sol^n. (i) $2x^2 - 6x + 3 = 0$

Here a=2,b=-6,c=3

Now b^2-4ac

$=(-6)^2 - 4.2.3$

$=36-24=12 > 0$

$\therefore b^2-4ac > 0$

$\therefore$ Roots are real and unequal .

Ex.2 For what value of p will the equation $x^2-2(5+2p)x+3(7+10p)=0$ have equal roots ?

Sol^n Here $a=1, b=-2(5+2p), c=3(7+10p)$

$\therefore$ for equal roots $b^2-4ac = 0$

$\{-2(5+2p)\}^2-4.1.3(7+10p)=0$

$\Rightarrow 4(5+2p)^2 - 4.3(7+10p)=0$

$\Rightarrow 4(25+4p^2+20p)-84-120p=0$

$\Rightarrow 100+16p^2+80p-84-120p=0$

$\Rightarrow 16p^2-40p+16=0$

$\Rightarrow 2p^2-5p+2=0$

$\Rightarrow 2p^2-4p-p+2=0$

$\Rightarrow 2p(p-2)-1(p-2)=0$

$\Rightarrow (p-2)(2p-1)=0$

$\therefore p=2, \frac{1}{2}$ Ans.

Ex.3 If α and β be the roots of $x^2 - 4x + 1 = 0$, Find the value of

(i) $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$ (iii) $\frac{1}{\alpha}+\frac{1}{\beta}$

Sol^n (i) given equation is $x^2 - 4x + 1 = 0$

Here a=1,b=-4 , c=1

Now $\alpha+\beta=\frac{-b}{a}=4$

$\therefore\ \alpha+\beta=4$

And $\alpha.\beta=\frac{c}{a}=1$

Now $(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha.\beta$

$\Rightarrow 4^2=\alpha^2+\beta^2+2.1$

$\Rightarrow 16\text{-}2=\alpha^2+\beta^2 \Rightarrow \alpha^2+\beta^2=14$

Sol^n (ii) $(\alpha+\beta)^3=\alpha^3+\beta^3+3\alpha.\beta(\alpha+\beta)$

$\Rightarrow 4^3=\alpha^3+\beta^3+3.1.4$

$\Rightarrow 64\text{-}12=\alpha^3+\beta^3 \Rightarrow \alpha^3+\beta^3=52$

Sol^n (iii) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{4}{1}=4$

Ex. 4 Find the condition that the roots of the equation $ax^2+bx+c=0$ may be in the ratio p:q .Prove that $Pqb^2=ac(p+q)^2$

Sol^n Let the root of the equation be $p\alpha$, $q\alpha$ so that the ratio is p:q

Sum of roots $= p\alpha+q\alpha=\frac{-b}{a}$

$=(p+q)\alpha=\frac{-b}{a}$ ………………(1)

and product of roots $= p\alpha.q\alpha=\frac{c}{a}$

$=p.q\alpha^2=\frac{c}{a}$ …………………..(2)

Eliminate α between (1) and (2) $\alpha=\frac{-b}{a(p+q)}$

Putting value of α in (2)

$Pq[\frac{-b}{a(p+q)}]^2=\frac{c}{a}$

$\Rightarrow Pq[\frac{b^2}{a(p+q)^2}]^2=\frac{c}{a}$

$\Rightarrow Pqb^2=ac(p+q)^2$

Ex.5 If the roots of the equation $x^2-mx+n=0$ be twice the other, prove that $2m^2=4n$.

Sol^n. Let the one root of the equation is α and other root is $\beta=2\alpha$.

$(\alpha+\beta)=(\alpha+2\alpha)=m$

$\Rightarrow 3\alpha=m$……………(1)

Again $\alpha.\beta=\alpha.2\alpha=n$

$\Rightarrow 2\alpha^2=n$………………..(2)

Eliminate α between (1) and (2) $\frac{m^2}{9}=\frac{n}{2}$

$\Rightarrow 2m^2=9n$ proved .

Ex.6 If the roots of the equation $ax^2+bx+c=0$ differ by p prove that $a(ap^2+4c)=b^2$.

Sol^n. Let the one root of the equation is α and other root is β

$$\alpha+\beta=\frac{-b}{a}$$

and $\alpha.\beta = \frac{c}{a}$, But $\alpha - \beta = p$

We have $(\alpha+\beta)^2 = (\alpha-\beta)^2 + 4\alpha.\beta$

$$\Rightarrow \left(\frac{-b}{a}\right)^2 = (p)^2 + 4\frac{c}{a}$$

$$\Rightarrow \frac{b^2}{a^2} = (p^2a+4c)/a$$

$$\Rightarrow a(ap^2+4c) = b^2. \text{ proved.}$$

Ex.7 Form an equation whose one root is $6+i\sqrt{7}$

Soln: We know, if p+ iq is a root of quadratic equation $ax^2+bx+c=0$, then the other root will be p- iq. Conversely if p- iq is a root of quadratic equation $ax^2+bx+c=0$, then the other root will be p+ iq .

So , $\because$ $6+i\sqrt{7}$ is a root therefore another root will be $6-i\sqrt{7}$

We also have , x^2 - (sum roots)x+ (product of roots) = 0

$$\Rightarrow x^2 - (6+i\sqrt{7} + 6 - i\sqrt{7})x + (6+i\sqrt{7})(6-i\sqrt{7}) = 0$$

$$\Rightarrow x^2 - 12x + (36 - i^2 7) = 0$$

$$\Rightarrow x^2 - 12x + (36+7) = 0$$

$$\Rightarrow x^2 - 12x + 43 = 0$$, which is the required equation .

Ex . 8 If the equations $x^2+px+q=0$ and $x^2+qx+p=0$, have a common root, then prove that either $p=q$ or $p+q+1=0$.

Soln: Let α be the common root of equations $x^2+px+q=0$ and $x^2+qx+p=0$

$\therefore$ $\alpha^2+p\alpha+q=0$ ……………..(i)

and $\alpha^2+q\alpha+p=0$ ……………(ii)

(i)-(ii) $\Rightarrow (p-q)\,\alpha + q - p = 0$

$\Rightarrow (p-q)(\alpha-1) = 0$

Either $p-q=0$ or $\alpha-1=0$

$\Rightarrow p=q$ or $\alpha=1$

Put $\alpha=1$ in (i) $1+p+q=0$ Proved .

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5. Arithmetic Progression

1.1 Arithmetic Progression :

In other words, in A.P series the difference between any term and its preceding one is constant .

The constant quantity is called the common difference(c.d)and it is denoted by d

Common difference = any term - preceding term

Note: The general representation of A.P series is a, a+d , a+2d , a+3d ,a+4d ....

Where $a = 1^{st}$ term , d = common difference

Here 1^{st} term = a

2^{nd} term = a+ d

3^{rd} term = a+ 2d

4^{th} term = a+ 3d

.
.
.
.

n^{th} term = a+ (n-1)d

Let t_n denotes the nth term . $\therefore$ t_n = a+ (n-1)d

Example of A.P Series (i) 2,4,6,8 .... Here , 4 -2 = 6 – 4 = 8 -6 =………= 2

1**.2 Arithmetic Mean(A.M) :** If the quantities a,m,b are inA.P then

$$m - a = b - m \Rightarrow m = \frac{a+b}{2}$$

i.e the arithmetic mean of a and b is m = $\frac{a+b}{2}$

Note: If the quantities a , m_1, m_2 , m_3 , m_4 ,……..m_n , b are inA.P then the intermediate numbers m_1, m_2 , m_3 , m_4 ,……..m_n are called the n arithmetic means of a and b .

1.3 Sum of A.P series upto n^{th} term.

Let a be the first term , d be the common difference and l be the last term of an A.P series.
Let S_n denote the sum

S_n = a+(a+d)+ (a+2d)+(a+3d)+…………. + (l-2d)+(l-d)+l …(i)

S_n = l+(l-d) + (l-2d) +………. + (a+d) + a …….(ii)

(i) + (ii) $\Rightarrow$ S_n = (a+l) + (a+l) +(a+l) + (a+l) = n(a+l)

$\Rightarrow$ $2S_n$ = n(a+l)

$\Rightarrow$ $S_n = \frac{n}{2}(a+l)$ here l is the n^{th} term $\therefore$ l={a+(n-1)d}

$= \frac{n}{2}$ {a+a+(n-1)d}

$\Rightarrow$ $S_n = \frac{n}{2}$ {2a + (n-1)d}

It gives the sum of A.P series up to nth term whose first term is a and common difference is d.

Worked out example .

Ex1. For the following series 7 , 10 ,13………..find the 13^{th} term.

Soln: We know nth term , t_n = a+ (n-1)d

Here , n = 13 , a = 7 , d = 10 -7 = 3

T_{13} = 7+ (13-1)3

= 7 + 36 = 43 $\therefore$ 13^{th} term is 43 Ans .

Ex2. Insert 7 arithmetic mean between 1 and 41 .

Soln: Let $m_1, m_2, m_3, m_4, m_5, m_6, m_7$ be 7 arithmetic mean between1and 41.

Then the series will be 1, $m_1, m_2, m_3, m_4, m_5, m_6, m_7$,41
Here total number of terms are 9 . $\therefore T_9 = 41$

Now , $T_9 = a + (9-1)d$ here a = 1

$\Rightarrow 41= 1 + 8d$

$\Rightarrow 40 = 8d$

$\Rightarrow d = 5$

$m_1 = a+d = 1+5 =6$

$m_2 = a +2d = 1+2.5 = 11$

$m_3 = a + 3d = 1+ 3.5 = 16$

$m_4 = a+4d = 1+4.5 = 21$

$m_5 = a+ 5d = 1+5.5 = 26$

$m_6 = a+6d = 1+6.5 = 31$

$m_7 = a+7d = 1+7.5 = 36$

$\therefore$ required arithmetic means are 6 ,11 , 16, 21, 26, 31, 36 Ans.

Ex 3. The 10^{th} term of an A.P series is 41 and 18^{th} term is 73 .Find the progression .

Soln: We know nth term , $t_n = a+ (n-1)d$

$\therefore$ 10^{th} term $t_{10} = a+ (10-1)d$

$41 = a+ 9d$ $(\because \; t_{10} = 41)$ …………….(i)

Again , 18^{th} term $t_{18} = a+ (18-1)d$

$73 = a+ 17d$ $(\because \; t_{18} = 73)$…………(ii)

(ii) – (i) $\Rightarrow$ 8d = 32 $\Rightarrow$ d = 4

Put d = 4 in (i) $41 = a+ 9.4 \Rightarrow a = 5$

We know the A.P series is a , a+d , a+2d , a+3d ,a+4d ....
i.e 5 , 5+4 = 9 , 5+2.4 = 13, 5+3.4 = 17, 5+4.4= 21,……….

i.e 5, 9, 13, 17, 21,………. Ans.

Ex 4. The 9^{th} term of an A.P series is 0 . Prove that 29^{th} term is double the 19^{th} term .

Soln: We know nth term , $t_n = a+ (n-1)d$

$\therefore$ 9^{th} term $t_9 = a+ (9-1)d$ $(\because \; t_9 = 0)$

$\Rightarrow 0 = a+8d$

$\Rightarrow a = -8d$

Now 19^{th} term, $t_{19} = a+ (19-1)d$

$= -8d+ 18d = 10d$

Again 29^{th} term, $t_{29} = a+ (29-1)d$

$= -8d + 28d$

$= 20d$

$=2.(10d)$

$=2.(t_{19})$ i.e $t_{29} = 2(t_{19})$ proved.

Ex 5. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P then prove that a^2, b^2, c^2 are in A.P

Soln: $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P $\therefore \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$$\Rightarrow \frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c+a)-(a+b)}{(a+b)(c+a)}$$

$$\Rightarrow \frac{(b-a)}{(c+a)(b+c)} = \frac{(c-b)}{(a+b)(c+a)}$$

$$\Rightarrow \frac{(b-a)}{(b+c)} = \frac{(c-b)}{(a+b)}$$

$$\Rightarrow (a+b)(b-a) = (c+b)(c-b)$$

$$\Rightarrow b^2 - a^2 = c^2 - b^2$$

$\Rightarrow a^2, b^2, c^2$ are in A.P proved.

Ex 6. In A.P $t_3 : t_5 = 1 : 4$; Show that $t_7 : t_{12} = 14 : 29$

Soln: We know nth term , $t_n = a + (n-1)d$

Given $t_3 : t_5 = 1 : 4$

$$\Rightarrow \frac{t_3}{t_5} = \frac{1}{4}$$

$$\Rightarrow \frac{a+(3-1)d}{a+(5-1)d} = \frac{1}{4}$$

$$\Rightarrow \frac{a+2d}{a+4d} = \frac{1}{4}$$

$\Rightarrow 4a+8d = a+4d \Rightarrow 3a = -4d \Rightarrow a = \frac{-4}{3}d$

Now, $\frac{t_7}{t_{12}} = \frac{a+6d}{a+11d}$

$$= \frac{\frac{-4d}{3}+6d}{\frac{-4d}{3}+11d}$$

$= \frac{-4d+18d}{-4d+33d} = \frac{14}{29}$ $\therefore \frac{t_7}{t_{12}} = \frac{14}{29}$ $\therefore t_7 : t_{12} = 14 : 29$ proved .

Ex 7. Find the sum of the following 5+8+11+……….upto 30th term .

Soln: We know $S_n = \frac{n}{2}[2a+(n-1)d]$

Here a=5 ,d= 3, n=30

$$S_n = \frac{30}{2}[2.5+(30-1)3]$$

$= 15\ [10+87] = 1445$

H.W

Ex1. If $t_n = 2n^2-2$ then find the series . Ans. 0 , 6 , 16 , 30……..

Ex 2. The 3rd term of an A.P series is 18 and 7th term is 30.Find the progression.
Ans. 12 , 15 , 18 , 21 , 24…………….

Ex 3. The 1st term of an A.P series is 6 and common difference is 2 Find the 15th term . Ans. 34

Ex 7. How many terms are there in the A.P series 7,13,19,25,……..,205? Ans. n = 34 term

Ex 8. (i) Insert 4 A.M between 2 and 12 . Ans . 4 , 6 , 8 , 10

(ii) Insert 3 A.M between -7 and 9 . Ans. -3 , 1 , 5

Ex 9. If a,b,c are in A.P then prove that

(i) $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are also in A.P.

(ii) (b+c) , (c+a) , (a+b) are also in A.P.

Ex 10. Find the sum of

(i) 3 +7 +11+…….+ 79 Ans. 820

(ii) 4 + 8 +12+…….+ 80 Ans. 840

****

6. Geometric Progression

1.1 The Geometric progression is a series of quantities when the ratio of any term(except the first) to the preceding one is constant . This constant is called the common ratio (c.r) and it is usually denoted by r.

The General form of G.P series is $a , ar , ar^2 , ar^3 ,$………………..

where $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$

i.e $\frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \ldots\ldots\ldots\ldots = \frac{a_{n+1}}{a_n}$ = constant for all $n \in N$

Here a is called the first term .

Now first term in G.P series is $(t_1) = a = ar^{1-1}$
Second term in G.P series is $(t_2) = ar = ar^{2-1}$
third term in G.P series is $(t_3) = ar^2 = ar^{3-1}$
⋮
⋮
nth term in G.P series is $(t_n) = ar^{n-1}$
If l is the last term then $l = ar^{n-1}$

Note:

1. If any term(except the first) is divided by the immediate preceding term, then common ratio can be determined .
2. If the first term is multiplied by the common ratio then 2^{nd} term can be obtained similarly if the 2^{nd} term is multiplied by the common ratio then 3^{nd} term can be obtained and similarly .

Example of G.P series : (i) 3 , 6 , 12 , 24 ,48…….
(ii) 1 ,4 ,16 , 64…………

Here in (i) $\frac{6}{3}=\frac{12}{6}=\frac{24}{12}=........=2$ i.e the ratio is constant throughout the series and is equal to two. Therefore in this case , common ratio is 2 .

Similarly in (ii) $\frac{4}{1}=\frac{16}{4}=$ …….= 4 i.e the ratio is constant throughout the series and is equal to four .Therefore in this case , common ratio is 4 .

1.2 Properties of G.P series :

1.If the term of G.P series are multiplied or divided by a constant then the products or quotients are also in G.P.

1.6.6 Geometric Mean (G.M) : Let three quantity a , G and b are in G.P . ThenG is said to be Geometric Mean of a and b

$$\text{if } \frac{G}{a}=\frac{b}{G} \Rightarrow G^2 = ab$$
$$\Rightarrow G = \sqrt{ab}$$

Note 1. The geometric mean between the two quantities is the square root of their product .

2. Again a and b be two given number . If G_1 , G_2 , G_3, …….G_n are inserted between a and b such that the sequence a , G_1 , G_2 , G_3, …….G_n , b is in G.P .

Then the numbers G_1 , G_2 , G_3, …….G_n are known as n geometric mean between a and b **.**

1.3 To Prove that A.M > G.M

Consider two number a and b .

Let A be the A.M between the number a and b then A = $\frac{a+b}{2}$

and G be the G.M between the number a and b then G = $\sqrt{ab}$

Now , A – G = $\frac{a+b}{2}$ - $\sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a})^2+(\sqrt{b})^2-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} > 0 \quad (\because a \neq b)$$

$\therefore$ A – G > 0

$\therefore$ A > G Proved.

1.4 Sum of G.P series :

Consider the General form of G.P series $a , ar , ar^2 , ar^3 ,\ldots\ldots\ldots$
where 1^{st} term is a and common ratio is r .

Let S_n denotes the sum of G.P series up to nth term .

$$S_n = a\frac{(1-r^n)}{1-r} \quad (\text{if } r<1)$$

For $r > 1$, $S_n = a\dfrac{(r^n-1)}{r-1}$

Note : When $r = 1$ then the G.P series becomes $a , a , a , a\ldots\ldots$ So
sum $= a+a+a+a+\ldots\ldots$ nth term $= na$

Now , Sum of G.P series up to infinity is denoted by S

$$\therefore S = \frac{a}{1-r} \quad \text{provided} \quad -1 < r <1$$

Worked out example :

Ex 1. Find 5^{th} term of 1 ,3 , 9 , 27…….

Sol^n : In the series 1 ,3 , 9 , 27……. First term $a= 1$ and common ratio $r = 3$
Here, $n = 5$
Now , n^{th} term in G.P series $a , ar , ar^2 , ar^3 ,\ldots\ldots$ is $t_n = ar^{n-1}$
$\therefore t_5 = 1.3^{5-1} = 3^4 = 81.$ Ans.

Ex 2. What term of the series 2,4,8,16……………is 2048 .
Sol^n : In the series 2,4,8,16…………….First term $a = 2$ and common ratio $r = 2$

Let, n^{th} term $(t_n) = 2048$

$\therefore ar^{n-1} = 2048$

$\Rightarrow 2.2^{n-1} = 2048$

$\Rightarrow 2^{n-1} = 1024$

$\Rightarrow \dfrac{2^n}{2} = 1024 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$ $\therefore$ 11^{th} term is 2048 .

Ex 3. The 4^{th} term of G.P series is 27 and 7^{th} term is 729. Find the G.P series .

Soln : If a is the first term and r is the common difference then

nth term in G.P series is $(t_n) = ar^{n-1}$
Here $n = 4$ and $t_4 = 27$ $\therefore 27 = ar^3$……………….(i)

Again $n = 7$ and $t_7 = 729$ $\therefore 729 = ar^6$……………….(ii)

$$\frac{(ii)}{(i)} \Rightarrow \frac{729}{27} = \frac{ar^6}{ar^3}$$

$\Rightarrow r^3 = 27 \Rightarrow r^3 = 3^3$

$\Rightarrow r = 3$ put $r = 3$ in (i) $27 = a3^3 \Rightarrow a = 1$

The General form of G.P series is $a , ar , ar^2 , ar^3 ,\ldots\ldots$
$\therefore$ Required G.P series is $1 , 1.3 , 1.3^2 ,\ldots\ldots$

$\Rightarrow$ 1 , 3 , 9………….. Ans .

4. Insert 3 geometric mean between 9 and $\frac{1}{9}$.

Soln: Let G_1, G_2, G_3 be three geometric mean between 9 and $\frac{1}{9}$.

Then the geometric series will be 9 , G_1, G_2, G_3 , $\frac{1}{9}$.

There are altogether 5terms . $\therefore$ $G_5 = ar^{5-1}$

Given , $G_5 = \frac{1}{9}$, a = 9 $\qquad \therefore \frac{1}{9} = 9r^4$

$\Rightarrow \frac{1}{81} = r^4 \Rightarrow r = \frac{1}{3}$

$\therefore$ $G_1 = ar = 9.\frac{1}{3} = 3$, $G_2 = ar^2 = 9.(\frac{1}{3})^2 = 1$ and $G_3 = ar^3 = 9.(\frac{1}{3})^3 = \frac{1}{3}$

$\therefore$ required G.M are 3 , 1 , $\frac{1}{3}$. Ans.

Ex 6. Insert 5 geometric mean between 576 and 9 .

Soln: Let G_1, G_2, G_3, G_4, G_5 be five geometric mean between 576 and 9 .

Then the geometric series will be 576 , G_1, G_2, G_3, G_4, G_5 , 9 .

There are altogether 7terms . $\therefore$ $G_7 = ar^{7-1}$

Given , $G_7 = 9$, a = 576 $\qquad \therefore 9 = 576r^4$

$\Rightarrow \frac{1}{64} = r^4 \Rightarrow r = \frac{1}{2}$

$\therefore$ $G_1 = ar = 576.\frac{1}{2} = 288$, $G_2 = ar^2 = 576.(\frac{1}{2})^2 = 144$, $G_3 = ar^3 = 576.(\frac{1}{2})^3 = 72$,

$G_4 = ar^4 = 576.(\frac{1}{2})^4 = 36$, $G_5 = ar^5 = 576.(\frac{1}{2})^5 = 18$

$\therefore$ required G.M are 288 , 144 , 72 , 36 , 18 . Ans.

Ex 7. If a,b ,c, d are in G.P then prove that a+b ,b+c ,c+d are also in G.P .

Soln: Let r be the common ratio then b=ar , c = ar^2 , d = ar^3

Now , a+ b = a + ar = a(1+ r)

b+ c = ar + ar^2 = ar (1 + r)

c + d = $ar^2 + ar^3 = ar^2(1 + r)$

Now , $\frac{b+c}{a+b} = \frac{ar(1+r)}{a(1+r)} = r$

Again $\frac{c+d}{b+c} = \frac{ar^2(1+r)}{ar(1+r)} = r$

$\therefore \frac{b+c}{a+b} = \frac{c+d}{b+c}$

$\therefore$ a+b ,b+c ,c+d are also in G.P .

h.w

Ex 1. The 4^{th} term of G.P series is $\frac{1}{12}$ and 8^{th} term is $\frac{1}{192}$. Find the 14^{th} term of G.P series .

Ans: $\frac{1}{12288}$

Ex 2. The 9^{th} term of G.P series is 64 and 15^{th} term is 4096 . Find the 11^{th} term of G.P series .

Ans: 256

Ex 3. The 5^{th} term of G.P series is 48 and 12^{th} term is 6144 . Find the 1^{st} and the common ratio .
Ans: First term a = 3 and common ratio r = 2

Ex 5. (i) Insert 3 geometric mean between $\frac{1}{3}$ and 432 Ans: 2 , 12 , 72

(ii) Insert 3 geometric mean between 1 and 256 . Ans: 4 , 16 , 64

(iii) Insert 6 geometric mean between 27 and $\frac{1}{81}$ Ans: $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

Ex 6. If 5, x , y , z ,405 are the first five terms of a G.P series then find the value of x , y , z .

Ans: 15 , 45 , 135 or -15 , 45 , -135

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7(a)
Permutation and Combination

An arrangement that can be formed by taking some or all of a finite set of things (or objects) is called a **Permutation**.
Order of the things is very important in case of permutation.
A permutation is said to be a **Linear Permutation** if the objects are arranged in a line. A linear permutation is simply called as a permutation. In other word ,
Permutation can be defined as the different arrangements which can be made out of a given set of things by taking some or all of them at a time .
Permutation of three letters P Q and R taking one , two three or four at a time are respectively,

P	PQ	PQR
Q	QR	QRP
R	RP	RPQ
	PR	PRQ
	RQ	QPR
	QP	RQP

1.1 Factorial Notation : The continuous product of the first 'n' natural numbers is called factorial n and is deonoted by n! or $\lfloor n$.

i.e, n! = 1×2×3x ….. ×(n-1) ×n.

$\lfloor n$ = n(n-1)(n-2)(n-3)(n-4)…………3.2.1
= n {(n-1)(n-2)(n-3)(n-4)…………3.2.1}

$= n\underline{|n-1}$

$\underline{|5} = 5.4.3.2.1 = 120$, $\underline{|3} = 3.2.1 = 6$; $\underline{|2} = 2.1 = 2$; $\underline{|1} = 1$

We have , $\underline{|n} = n\underline{|n-1}$

$$n = \frac{\underline{|n}}{\underline{|n-1}} \quad \text{Put } n = 1\text{, we have, } 1 = \frac{\underline{|1}}{\underline{|1-1}}$$

$$\therefore\ 1 = \frac{1}{\underline{|0}}$$

So we define $\underline{|0} = 1$

The number of permutation of n distinct objects taken r at a time ($1 \le r \le n$)
is generally denoted by ${}^{n}P_{r}$. or p(n,r) .

$${}^{n}P_{r} = n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-r+1) = \frac{\underline{|n}}{\underline{|n-r}} \ldots\ldots\ldots\ldots\text{(i)}$$

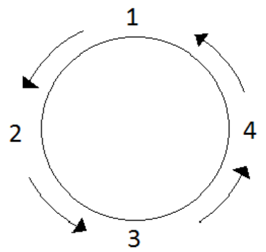
In (i) , if r = n we have , $\underline{|n} = n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots3.2.1$

Eg. 3 favourite desserts, in order from a menu of 10 is ${}^{10}P_{3} = 720$.

1.2 Cor: (i) The number of permutation of n different things taken at a time is ${}^{n}P_{n} = \underline{|n}$

$$\text{(ii)}\ {}^{n}P_{0} = \frac{\underline{|n}}{\underline{|n-0}} = \frac{\underline{|n}}{\underline{|n}} = 1$$

1.3 Circular permutation : We have discussed about linear permutation . In linear permutation , things are arranged in a row ,but in **Circular permutation** things are arranged in a circle where there is no end and no beginning . For example
Consider , four numbers 1, 2 , 3, 4 .



Starting from each number ,we have the arrangement
1234, 2341,3412, 4123(anticlockwise)
Thus corresponding to 4 linear arrangements , there is 1 circular arrangement .($\frac{4}{4} = 1$)

But 4 numbers can be arranged in a straight line in $\underline{|4}$ ways .

$\therefore$ 4 number we can arranged in a circle is $\frac{\underline{|4}}{4}$

1.4 Cor : n different things , we can arranged in a circle is $\frac{\underline{|n}}{n}$ ways .

1.5 Prove that ${}^{n}P_{r} = {}^{n-1}P_{r} + r.\ {}^{n-1}P_{r-1}$

Proof: ${}^{n-1}P_{r} + r.\ {}^{n-1}P_{r-1} = \frac{\underline{|n-1}}{\underline{|n-1-r}} + r.\frac{\underline{|n-1}}{\underline{|(n-1)-(r-1)}}$

$$= \frac{\underline{|n-1}}{\underline{|n-1-r}} + r.\frac{\underline{|n-1}}{\underline{|n-r}}$$

$$= \frac{\underline{|n-1}}{\underline{|n-1-r}} + r.\frac{\underline{|n-1}}{(n-r)\underline{|n-r-1}}$$

$$= \frac{\underline{|n-1}}{\underline{|n-1-r}}\left[1+\frac{r}{(n-r)}\right] = \frac{\underline{|n-1}}{\underline{|n-1-r}}\left[\frac{n-r+r}{n-r}\right]$$

$$= \frac{\underline{|n-1}}{\underline{|n-1-r}}\left[\frac{n}{n-r}\right] = \frac{\underline{|n}}{\underline{|n-r}} = {}^{n}P_{r} \quad \text{Proved .}$$

Worked out example:

Ex. 1 Find value of (i) $^6P_3$ (ii) $^8P_0$

Soln : (i) $^6P_3 = \dfrac{\lfloor 6}{\lfloor 6-3} = \dfrac{\lfloor 6}{\lfloor 3} = \dfrac{6.5.4.3.2.1}{3.2.1} = 5.4.3.2.1 = 120$ Ans.

(ii) $^8P_0 = \dfrac{\lfloor 8}{\lfloor 8-0} = \dfrac{\lfloor 8}{\lfloor 8} = 1$ Ans .

Ex. 2 Find n if $^nP_4 = 10\times\ ^nP_3$

Soln: Given , $^nP_4 = 10\times\ ^nP_3$

$$\Rightarrow \frac{n!}{(n-4)!} = 10\times\frac{n!}{(n-3)!}$$

$$\Rightarrow \frac{n!}{(n-4)!} = 10\times\frac{n!}{(n-3)(n-4)!}$$

$$\Rightarrow 1 = 10\times\frac{1}{(n-3)}$$

$$\Rightarrow n - 3 = 10 \Rightarrow n = 13 \text{ Ans.}$$

Ex.3 If $^{n+1}P_6 : {}^{n-1}P_7 = 5 : 12$ then find the value of n.

Soln: We have , $^{n+1}P_6 = \dfrac{\lfloor n+1}{\lfloor n+1-6}$ and $^{n-1}P_7 = \dfrac{\lfloor n-1}{\lfloor n-1-7}$

Now , $\dfrac{^{n+1}P_6}{^{n-1}P_7} = \dfrac{5}{12}$

$$\Rightarrow \frac{\dfrac{\lfloor n+1}{\lfloor n+1-6}}{\dfrac{\lfloor n-1}{\lfloor n-1-7}} = \frac{5}{12}$$

$$\Rightarrow \frac{\dfrac{\lfloor n+1}{\lfloor n-5}}{\dfrac{\lfloor n-1}{\lfloor n-8}} = \frac{5}{12}$$

$$\Rightarrow \frac{n(n+1)}{(n-5)(n-6)(n-7)} = \frac{5}{12}$$

or , $12n(n+1) = 5(n-5)(n-6)(n-7)$

or , $5n^3 - 102n^2+523n -1050 = 0$

or $(n-14)(5n^2-32n+75) = 0$

Either $n-14 = 0$ or $5n^2-32n+75 = 0$

But any real value of n does not satisfy the equation $5n^2-32n+75 = 0$ therefore $n = 14$ Ans .

Ex.5 prove that $1.{}^1P_1+2.{}^2P_2+3.{}^3P_3+\ldots\ldots\ldots\ldots+n.{}^nP_n = {}^{n+1}P_{n+1}-1$

Soln. L.h.s. $1.{}^1P_1+2.{}^2P_2+3.{}^3P_3+\ldots\ldots\ldots\ldots+n.{}^nP_n$

$= (2-1).{}^1P_1+(3-1).{}^2P_2+ (4-1).{}^3P_3+\ldots\ldots\ldots\ldots+ [(n+1)-1]\,{}^nP_n$

$= 2.{}^1P_1- 1.{}^1P_1+ 3.{}^2P_2 -1.{}^2P_2+ 4.{}^3P_3 -1.{}^3P_3 +\ldots\ldots\ldots\ldots+ (n+1)\,{}^nP_n- 1.\,{}^nP_n$

$= {}^2P_2 - {}^1P_1 + {}^3P_3- {}^2P_2+ {}^4P_4 - {}^3P_3\ldots\ldots\ldots\ldots\ldots+ {}^{n+1}P_{n+1}- {}^nP_n$

[Since , $(r+1)\,{}^rP_r = (r+1)(r)! = (r+1)! = {}^{r+1}P_{r+1}$

$= {}^{n+1}P_{n+1} - {}^{1}P_{1}$
$= {}^{n+1}P_{n+1} - 1$ Proved.

Ex.6 In how many ways can the letters of the word "MATHEMATICS" be arranged .
Soln: In the word "MATHEMATICS" there are altogether 11 letters .
Out of these 11 letters , "M" occurs 2 times , "A" occurs 2 times , "T" occurs 2 times .

So the total number of possible arrangement $= \dfrac{\lfloor 11}{\lfloor 2 \times \lfloor 2 \times \lfloor 2} = 4989600$.

Ex.7 In how many ways the letter of the word DAUGHTER be arranged so that the vowls may never be separated ?

Soln: In the word DAUGHTER, we will take three vowel A , U, E as one letter .
Other letters are D,G,H,T,R . therefore altogether total letters are DGHTR(AUE) i.e 6 letters to arrange .
Therefore number of arrangement (keeping the vowels together) ${}^{6}P_{6} = \lfloor 6 = 720$.
Again vowels can be arranged (among themselves) in $\lfloor 3$ ways . = 6
Therefore total number of arrangements $720 \times 6 = 4320$. Ans

Ex.8 A child has 3 pockets and 4 coins. In how many ways can he put the coins in his pocket.

Ans. First coin can be put in 3 ways, similarly second, third and forth coins also can be put in 3 ways.

So total number of ways = 3 x 3 x 3 x 3 $= 3^4 = 81$

Ex. 9 In how many ways can the letters of the word "EDUCATION" be arranged .
Soln. In "EDUCATION" there are 9 letters .
Therefore required arrangement = $\lfloor 9$ Ans .

Ex.10 In how many ways can 6 boys form a ring ?

Soln. Let the boys be A, B ,C, D,E ,F
Let A stand up in certain position . Then the remaining 5 can be arranged in $\lfloor 5$ ways among themselves .
Therefore required number of arrangement $= \lfloor 5 = 120$. Ans.

H.W

Ex.1 Find n if

(i) ${}^{n}P_{4} = 12 \times {}^{n}P_{2}$ Ans. (i) n = 6
(ii) ${}^{n}P_{4} = 10 \times {}^{n-1}P_{3}$ Ans .(ii) n= 10
(iii) ${}^{n}P_{5} = 90 \times {}^{n-2}P_{3}$ Ans .(iii) n= 10

Ex. 2 Find the value of n if

(i) ${}^{n-1}P_{3} : {}^{n+1}P_{3} = 5 : 12$ (i) Ans. n = 8 , $\dfrac{9}{7}$
(ii) ${}^{n}P_{4} : {}^{n+1}P_{4} = 5 : 9$ (ii) Ans. n = 8

Ex.3 If ${}^{n+r}P_{2} = 90$ and ${}^{n-r}P_{2} = 30$ find n and r . Ans. n = 8 and r = 2

Ex. 4 In how many ways can the letters of the word "COLLEGE" be arranged . Ans 1260 .

Ex.5 In how many ways can the letters of the word "JALPAIGURI" be arranged .
Ans.9,07,200

Ex. 6 In how many ways can the letters of the word "SUNDAY" be arranged . Ans.720

Ex. 7 In how many ways can 8 boys form a ring ?

Ans. 8.

Ex. 8 In how many ways can 16 boys form a ring ?

7(B)
Combination :

A selection that can be formed by taking some or all of a finite set of things(or objects) is called a **Combination**

The number of combinations of n dissimilar things taken r at a time is denoted by ${}^nC_r$ or C(n , r)

1.1 Theorem : The number of combination of n different things taken r at a time is

$${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$

Proof: Let ${}^nC_r$ be the required number of combinations . Since every combination of r different things produces $\underline{|r}$ permutations when the r things are arranged in all possible ways , ${}^nC_r$ combinations , each combinations containing r different things ,will produce ${}^nC_r \times \underline{|r}$ permutations . Again number of permutations of n different things taken r at a time is ${}^nP_r$

Hence , ${}^nC_r \times \underline{|r} = {}^nP_r$

$$= \frac{\underline{|n}}{\underline{|n-r}}$$

$$\therefore \quad {}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$

Eg. (i) Choosing 3 desserts from a menu of 10 is ${}^{10}C_3 = 120$.

(ii) Picking a team of 4 people from a group of 12 = ${}^{12}C_4 = 495$

1.2 Deduction : (i) ${}^nC_1 = \dfrac{\underline{|n}}{\underline{|1}\,\underline{|n-1}} = \dfrac{n\underline{|n-1}}{\underline{|n-1}} = n$

(ii) ${}^nC_n = \dfrac{\underline{|n}}{\underline{|n}\,\underline{|n-n}} = \dfrac{\underline{|n}}{\underline{|n}\,\underline{|0}} = 1$ $(\underline{|0} = 1)$

(iii) Prove that ${}^nC_{n-r} = {}^nC_r$

Proof : ${}^nC_{n-r} = \dfrac{\underline{|n}}{\underline{|n-r}\,\underline{|n-n+r}} = \dfrac{\underline{|n}}{\underline{|n-r}\,\underline{|r}} = {}^nC_r$

$\therefore$ ${}^nC_{n-r} = {}^nC_r$ Proved.

(iv) Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(v) ${}^nC_r \times \underline{|r} = {}^nP_r$

Worked out example :

Ex.1 Find the value of ${}^7C_5$

Soln. ${}^7C_5 = \dfrac{\underline{|7}}{\underline{|5}\,\underline{|7-5}} = \dfrac{7.6\underline{|5}}{\underline{|5}\,\underline{|2}} = \dfrac{7.6}{2.1} = 21$ Ans.

Ex.2 If ${}^nP_r = 110$, ${}^nC_r = 55$ find r .

Soln: We have , $^nC_r \times \lfloor r = {}^nP_r$

$\Rightarrow 55 \times \lfloor r = 110$

$\Rightarrow \lfloor r = \dfrac{110}{55} = 2 = 2.1 = \lfloor 2$

$\Rightarrow r = 2$ Ans.

Ex. 3 If $^{2n}C_3 : {}^nC_2 = 44 : 3$ find n .

Soln. $\dfrac{^{2n}C_3}{^nC_2} = \dfrac{44}{3}$

$$\Rightarrow \dfrac{\dfrac{\lfloor 2n}{\lfloor 3 \lfloor 2n-3}}{\dfrac{\lfloor n}{\lfloor 2 \lfloor n-2}} = \dfrac{44}{3}$$

$$\Rightarrow \dfrac{\dfrac{2n(2n-1)(2n-2)\lfloor 2n-3}{3.2.1\lfloor 2n-3}}{\dfrac{n(n-1)\lfloor n-2}{2.1\lfloor n-2}} = \dfrac{44}{3}$$

$$\Rightarrow \dfrac{\dfrac{n(2n-1)(2n-2)}{3}}{\dfrac{n(n-1)}{2}} = \dfrac{44}{3}$$

$$\Rightarrow \dfrac{n(2n-1)(2n-2)2}{3n(n-1)} = \dfrac{44}{3}$$

$\Rightarrow 8n^2 - 12n + 4 = 44n - 44$

$\Rightarrow 8n^2 - 56n + 48 = 0$

$\Rightarrow n^2 - 7n + 6 = 0$

$\Rightarrow (n-6)(n-1) = 0$

$\Rightarrow n = 6$ or $n = 1$

But n = 1is impossible $\therefore$ n = 6 Ans .

Ex.6 In how many ways 4 pencil can be drawn from a box containing 10 pencils ?

Soln. 4 pencil can be drawn from a box containing 10 pencils is equal to number of combination of 10 different things taken 4 at a time is

$$^{10}C_4 = \dfrac{\lfloor 10}{\lfloor 4 \lfloor 10-4} = 210 \quad \text{Ans.}$$

Ex.9 In how many ways 12 different books can be distributed equally among 3 students ?

Soln. Each student will get 4 books .

The first student may get 4 books in $^{12}C_4$ ways .

After it is done the student may get 4 books in $^8C_4$ ways .

The remaining 4 books may be given to the third student(in one way)

Thus required number of ways $^{12}C_4 \times {}^8C_4 = 34650$ Ans.

HW

Ex.1 Find the value of

(i) $^{10}C_3$ (ii) $^{15}C_{15}$ (iii) $^{16}C_{15}$

Ans: (i) 120 (ii) 1 (iii) 16

Ex.2

(a) If $^{15}C_r = {}^{15}C_{r-3}$ find r . Ans. (a) 9

(b) If $^{n}C_{15} = {}^{n}C_{10}$ find n Ans.(b) 25

Ex. 3

(a) If $^{n}C_{10} = {}^{n}C_3$ find $^{n}C_2$. Ans. (a)78

(b) If $^{20}C_r = {}^{20}C_{2r-1}$ find $^{10}C_r$. Ans. (b) 120

(c) If $^{20}C_r = {}^{20}C_{2r-1}$ find $^{r}C_5$. Ans. (c)21

Ex. 4 If $^{2n}C_4 : {}^{n}C_3 = 35 : 2$ find n . Ans. 4

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8 .Determinants

Formation:

First we will discuss how a determinant forms.

Let us consider two linear equation

$a_1x+b_1y = d_1$……………….(1)

$a_2x+b_2y = d_2$……………......(2)

$(1)\times b_2 \Rightarrow b_2a_1x + b_2b_1y = b_2d_1$ ………………...(3)

$(2) \times b_1 \Rightarrow b_1a_2x+ b_1b_2y = b_1d_2$….…………………(4)

$(3) - (4) \Rightarrow (b_2a_1 - b_1a_2)x = b_2d_1 - b_1d_2$

$$\Rightarrow x = \frac{b_2d_1 - b_1d_2}{b_2a_1 - b_1a_2}$$

Again , $(1) \times a_2 \Rightarrow a_2a_1x + a_2b_1y = a_2d_1$……………….(5)

$(2) \times a_1 \Rightarrow a_2a_1x + a_1b_2y = a_1d_2$………………(6)

$(5) - (6) \Rightarrow (a_2b_1 - a_1b_2)y = a_2d_1 - a_1d_2$

$$\Rightarrow y = \frac{a_1d_2 - a_2d_1}{b_2a_1 - b_1a_2}$$

The common denominator $(b_2a_1 - b_1a_2)$ can be written as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Ie. $(b_2a_1 - b_1a_2) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ …………………(I)

Right hand side of (I) is called determinant of 2nd order .

Now let us take three linear equation $a_1x+b_1y+c_1z = 0$

$a_2x+b_2y+c_2z = 0$

$a_3x+b_3y+c_3z = 0$

We find a relation from these nine co-efficient if the three equation are satisfied by the same value of x,y,z . So from 2nd and 3rd equation applying cross multiplication, we get,

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{b_3a_2 - a_3b_2}$$

Substituting the proportionate value of x ,y , z in the first equation we have
$a_1 (b_2 c_3 – b_3 c_2) + b_1 (c_2 a_3 – c_3 a_2) + c_1(a_2 b_3 – a_3b_2) = 0$……………….(12)

If we arrange all these co-efficient in successive rows then it will be

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$$

With this arrangement , the result shown in (12) can be symbolically written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

this is a 3rd order determinant because it has three rows and three columns . when expanded we will get left side of (12)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (b_2 c_3 – b_3 c_2) + b_1 (c_2 a_3 – c_3 a_2) + c_1(a_2 b_3 – a_3b_2)$$

$$= a_1 (b_2 c_3 – b_3 c_2) – b_1 (a_2 c_3 – a_3 c_2) + c_1(a_2 b_3 – a_3b_2) \text{………}(13)$$

Minor AND CO-Factor : In the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the minor of any element is the determinant obtained from Δ , without changing the order of elements by omitting the row and column containing that element .

For example minor of a_1 in Δ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ here the 1st column and 1st row which contains a_1 is omitted .Similarly we get the minor of $b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, minor of $c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

and so on. Thus minor of all factor can be determined .

The co-factor of any element a_{ij} is denoted by c_{ij} and is defined by $c_{ij} = (-1)^{i+j} M_{ij}$

Where is the M_{ij} minor of a_{ij} .

Co-factor of a_1 in $\Delta = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ = minor of a_1

CO-factor of $b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ = - minor of b_1

CO-factor of $c_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ = minor of c_1 and so on .

Ex. Find the minor and cofactor of the element 4 and 0 in the determinant

$$\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 4 & 1 & -2 \end{vmatrix}$$

Soln : The minor of 4 = $\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$ =1.2-0.3 =2

and cofactor of 4 = $(-1)^{1+3}$minor of 4 (4 lies in 1st column and 3rd row)

$= (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$ =1.2 = 2

The minor of 0 = $\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}$ = 2.(-2) – 3.4 = - 4 – 12 = -16

and cofactor of 0 = $(-1)^{2+2}$minor of 0 (0 lies in 2nd column and 2nd row)

$= (-1)^4 \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}$ = -16

Laplace's Expansion: A determinant can be Expanded in terms of row or column as follows :

For the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1 (b_2 c_3 – b_3 c_2) – b_1 (a_2 c_3 – a_3 c_2) + c_1(a_2 b_3 – a_3 b_2)$

Properties of Determinant :

The following properties of determinant is same for any order .

1. If every element of a row (or column) of a determinant is zero ,then determinant vanishes .

$$\Delta = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0$$

For ex, $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ -3 & 0 & 2 \\ 4 & 1 & -2 \end{vmatrix} = 0$

2. The value of the determinant remains unchanged if its rows and columns are interchanged .

For$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Interchanging rows and columns we have $\Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

When expanding $\Delta_1 = \Delta_2$.

3. If two rows (or columns) of a determinant are interchangedthen its numerical value remain same ,but sign is changed .

For $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then when expanding

we get $\Delta_1 = -\Delta_2$.

4. If each element of a row(or column) of a determinant is multiplied by a constant 'c' then its value gets multiplied by 'c' .

Ie. $\begin{vmatrix} ca_1 & cb_1 & cc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

5. If two rows (or columns) of a determinant is identical then the value of determinant is zero .

$\begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_1 & b_3 \\ c_1 & c_1 & c_3 \end{vmatrix} = 0$ (here 1st and 2nd column of the determinant are same)

(on expansion we will get value 0)

Addition of a determinants :

Theorem 1 : If every element in any row (or column) consists of the sum (or difference) of two quantities then the determinant can be expressed as the sum (or difference) of determinants of same order .

As $\begin{vmatrix} a_1+\alpha_1 & b_1 \\ a_2+\alpha_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix}$

Similarly, $\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$

Theorem 2 . If value of determinant remains unaltered by adding(or subtracting) to all the element of any particular row (or column) the same multiples of corresponding elements of one or more other rows(or column) .

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1+\alpha b_1 & b_1 & c_1 \\ a_2+\alpha b_2 & b_2 & c_2 \\ a_3+\alpha b_3 & b_3 & c_3 \end{vmatrix}$

Now $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha b_1 & b_1 & c_1 \\ \alpha b_2 & b_2 & c_2 \\ \alpha b_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \alpha \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \alpha .0$$

(Since two columns of the determinant are same so value of determinant is zero .)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \square$$

Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$

Some important important terms :

Transpose Determinant: Let $\square = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

By changing the rows and column we have $\square_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ Then $\square_1$ is called transpose of $\square$.

Adjoint of determinant: The determinant obtained by replacing each element of a given determinant by the co-factor is known as adjoint determinant

Cramers Rule :

Consider the linear equations

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

The solution of this equations with the help of determinant are (At present stage details steps are not shown.)

$$x = \frac{1}{\Delta}\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad y = \frac{1}{\Delta}\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad z = \frac{1}{\Delta}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note: Here, $\Delta \neq 0$ if $\Delta = 0$ then Cramers Rule is not applicable .

Worked out example :

Ex1. Find the value of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 2 & 4 & 6 \end{vmatrix}$

Soln : $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 2 & 4 & 6 \end{vmatrix}$ = 1(5.6-4.7) - 2(3.6 – 2.7) + 3(3.4 – 5.2)

= (30 – 28) – 2.(18 -14) + 3(12- 10)
= -2 -2(- 4)+3(2)
= -2 +8 + 6
=12

Ex2. Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Soln : Now , $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$ (replace first column by $c_1+c_2+c_3$)

= 0 (by prop.1 ,If every element of a row (or column) of a determinant is zero ,then determinant vanishes.)

Ex3. Evaluate $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Soln : $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

$= \begin{vmatrix} 1+w+w^2 & w & w^2 \\ 1+w+w^2 & w^2 & 1 \\ 1+w+w^2 & 1 & w \end{vmatrix}$ replace first column by $c_1+c_2+c_3$)

$= \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix}$ $\because\ 1+w+w^2 = 0$

= 0 (by prop.1 ,If every element of a row (or column) of a determinant is zero ,then determinant vanishes.)

Ex 4. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc+ab+bc+ca$$

Soln :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a & 0 & 1 \\ -b & b & 1 \\ 0 & -c & 1+c \end{vmatrix} \qquad C_1 \leftarrow C_1 - C_2 \text{ and } C_2 \leftarrow C_2 - C_3$$

$$= a\begin{vmatrix} b & 1 \\ -c & 1+c \end{vmatrix} - 0 + 1\begin{vmatrix} -b & b \\ 0 & -c \end{vmatrix}$$

= a {b(1+c)} – 1.(-c)} + {(-b)(-c) -0.b}

= a{b+bc +c} + { bc}

= ab+abc+ac+bc

= abc+ab+bc+ca R.H.S . proved.

Ex 5. Prove that $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix}$ = ab(a -1)(b -1)(b - a)

Soln: $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix} = ab \begin{vmatrix} 1 & 1 & 1 \\ a & b & 1 \\ a^2 & b^2 & 1 \end{vmatrix}$

$$= ab\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & 1-a \\ a^2 & b^2-a^2 & 1-a^2 \end{vmatrix} \qquad C_2 \leftarrow C_2 - C_1 \text{ and } C_3 \leftarrow C_3 - C_1$$

$$= ab(b-a)(1-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & 1+a \end{vmatrix}$$

$$= ab(b-a)(1-a)\begin{vmatrix} 1 & 1 \\ b+a & 1+a \end{vmatrix}$$

= ab(b - a)(1- a) {1+a – b - a}

= ab(b - a)(1- a)(1- b)

= ab(b - a)(a- 1)(b - 1) R.H.S. proved.

Ex 7. Prove that $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Soln: $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \dfrac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$ (multiplying 1st 2nd 3rd rows by a, b,cespectively.)

$$= \frac{1}{abc}\,\text{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad \text{Proved .}$$

Ex 8. Prove that $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$ is a perfact square .

Soln: $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$

$$=(a^2+2ab+b^2)\begin{vmatrix} 1 & 2ab & b^2 \\ 1 & a^2 & 2ab \\ 1 & b^2 & a^2 \end{vmatrix} \qquad C_1 \leftarrow C_1 + C_2 + C_3$$

$$= (a+b)^2\begin{vmatrix} 0 & 2ab-a^2 & b^2-2ab \\ 0 & a^2-b^2 & 2ab-a^2 \\ 1 & b^2 & a^2 \end{vmatrix}$$

$$= (a+b)^2\begin{vmatrix} 2ab-a^2 & b^2-2ab \\ a^2-b^2 & 2ab-a^2 \end{vmatrix}$$

$= (a+b)^2\{(2ab-a^2)^2 - (a^2-b^2)(b^2-2ab)\}$

$= (a+b)^2 (a^2+b^2 - ab)^2$

$= \{(a+b)(a^2+b^2 - ab)\}^2$

$= (a^3 - b^3)^2$ Which is a perfect square.

Ex 9. Solve by cramers rule $2x + 3y + 4z = 0$
$x + y + z = 0$
$2x - y + 3z = 0$

Soln: Let $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 2\{1.3-1(-1)\}-3(1.3-2.1) + \{1.(-1)-2.1\}$

$= 2\{3+1\}- 3(3-2) + 4(-1 - 2)$

$= 2\{4\}- 3 - 12$

$= -7 \neq 0$

$\therefore \quad \Delta = -7$

$$\Delta_1 = \begin{vmatrix} 0 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 0 \ (\because \text{ first column of } \Delta_1 \text{ is } 0 .)$$

$$\Delta_2 = \begin{vmatrix} 2 & 0 & 4 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 0 \ (\because \text{ second column of } \Delta_1 \text{ is } 0 .)$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix} = 0 \ (\because \text{ third column of } \Delta_1 \text{ is } 0 .)$$

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{-7} = 0 \ ; \quad y = \frac{\Delta_2}{\Delta} = \frac{0}{-7} = 0 \ ; \quad z = \frac{\Delta_3}{\Delta} = \frac{0}{-7} = 0 \ ;$$

$\therefore$ x = 0 ; y = 0 ; z = 0 ; Ans .

H.W

1. Evaluate

(i) $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ Ans : (i) 0 , (ii) - 20 ,

2. Prove by using properties .

(i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (iii) $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$

Ex3. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ is a perfact square .

Ex4 . (i) Solve $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x+\alpha & \beta & \gamma \\ \beta & x+\beta & \alpha \\ \alpha & \beta & x+\gamma \end{vmatrix} = 0$

Ans: (i) 1 , 1 , -2 (ii) 0 ,0 ,-(a+b+c)

Ex5. Using property prove the following .

(i) $\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$

Ex 6. Solve by cramers rule

(i) $3x + y + z = 0$, $x - 4y + 3z = 0$, $2x + 5y - 2z = 0$

(ii) $5x - y = 9$, $3x + y = 7$, $x + y + z = 4$

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Trigonometry

2

Trigonometric ratios and associated angle

2.1. Definition of Trigonometric function :

In the right angle triangle ΔOPQ, $\angle$OQP $=90^o$ $\angle$POQ $= \theta$, $\angle$OPQ $= 90^o - \theta$; OQ is base , PQ is altitude and OP is hypotenuse .

The trigonometric functions means the six trigonometric ratios.

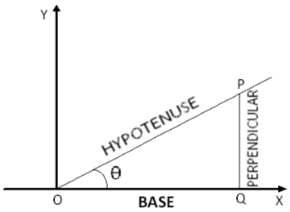

Fig. 2.2.1

2.2 Trigonometric ratios with the help of right angled triangle (fig 2.1)

the six trigonometric ratios are :

$\frac{OQ}{OP} = \frac{Base}{hypotenuse}$ is called cosine of θ and is written as $\cos\theta$ $\therefore \frac{OQ}{OP} = \cos\theta$

$\frac{PQ}{OP} = \frac{altitude}{hypotenuse}$ is called sine of θ and is written as $\sin\theta$ $\therefore \frac{PQ}{OP} = \sin\theta$

$\frac{PQ}{OQ} = \frac{altitude}{base}$, is called tangent of θ and is written as $\tan\theta$ $\therefore \frac{PQ}{OQ} = \tan\theta$

$\frac{OP}{PQ} = \frac{hypotenuse}{altitude}$ is called cosine of θ and is written as cosecθ $\therefore \frac{OP}{PQ} = \cos ec\theta$

$\frac{OP}{OQ} = \frac{hypotenuous}{base}$ is called secant of θ and is written as secθ $\therefore \frac{OP}{OQ} = \sec\theta$

$\frac{OQ}{PQ} = \frac{Base}{Altitude}$ is called cotangent of θ and is written as cotθ . $\therefore \frac{OQ}{PQ} = \cot\theta$

Note : (i) Sin$\theta \neq$ Sin $\times\ \theta$, Sin $\times\ \theta$ is meaningless Similarly Cos$\theta \neq$ Cos $\times\ \theta$

2.3. Relation between trigonometric ratios and function **:** From the triangle ΔPOQ with the help of Pythagoras theorem

$$PQ^2 + OQ^2 = OP^2$$

Divide by OP^2

$$\left(\frac{PQ^2}{OP^2}\right) + \left(\frac{OQ^2}{OP^2}\right) = 1$$

$$\text{or} \quad \left(\frac{PQ}{OP}\right)^2 + \left(\frac{OQ}{OP}\right)^2 = 1$$

$$\text{or} \quad (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\text{or} \quad \sin^2\theta + \cos^2\theta = 1$$

Again, $PQ^2 + OQ^2 = OP^2$

Divide by PQ^2

$$\left(\frac{PQ^2}{PQ^2}\right) + \left(\frac{OQ^2}{PQ^2}\right) = \left(\frac{OP^2}{PQ^2}\right)$$

$$\text{or} \quad \left(\frac{PQ}{PQ}\right)^2 + \left(\frac{OQ}{PQ}\right)^2 = \left(\frac{OP}{PQ}\right)^2 \quad \text{or} \quad 1+\cot^2\theta = \operatorname{cosec}^2\theta$$

Similarly we can prove $\tan^2\theta + 1 = \sec^2\theta$

Note : Therefore we have

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1+\cot^2\theta = \operatorname{cosec}^2\theta$
3. $\tan^2\theta + 1 = \sec^2\theta$

Note: $\sin^2\theta$ and $\sin\theta^2$ have different meaning .

Here we will express all trigonometrical ratios in **cosine ratios** .

From (1) we have $\sin^2\theta + \cos^2\theta = 1$

or . $\sin^2\theta = 1 - \cos^2\theta$

or $\sin\theta = \sqrt{1-\cos^2\theta}$

Worked out example:

Ex1. Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Sol^n :

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{(1-\sin^2\theta)}}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta \quad \text{Proved.}$$

Ex2. Prove that $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

Soln:

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} + \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1-\cos\theta+1+\cos\theta}{\sin\theta}$$

$$= \frac{2}{\sin\theta} = 2\text{cosec}\theta. \quad \text{Proved.}$$

Ex3. Prove that $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta.\cos\theta$

Soln: $(\sin\theta - \cos\theta)^2 = \sin^2\theta + \cos^2\theta - 2\sin\theta.\cos\theta = 1 - 2\sin\theta.\cos\theta$ Proved.

Ex4. Prove that $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Soln: $= \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}$

$$= \frac{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \cos^2\theta-\sin^2\theta \quad \text{Proved.}$$

Ex.5. Prove that $(1+\tan^2A)(1-\sin^2A) = 1$

Soln: $(1+\tan^2A)(1-\sin^2A)$

$= \sec^2A.\cos^2A$

$= \dfrac{1}{\cos^2A}.\cos^2A = 1$ Proved.

Ex.6. Prove that $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$

Sol^n: L.H.S $= \sec^4\theta - \sec^2\theta = \sec^2\theta(\sec^2\theta - 1)$

$= \sec^2\theta\tan^2\theta$

$= (1+\tan^2\theta)\tan^2\theta$

$= \tan^2\theta + \tan^4\theta$. Proved.

Ex 7. If $\tan\theta = c$ then prove that $\sin\theta = \dfrac{c}{\sqrt{1+c^2}}$

Soln: Given, $\tan\theta = c$ $\therefore$ $\cot\theta = \dfrac{1}{c}$, we have $1 + \cot^2\theta = \text{cosec}^2\theta$

$\therefore$ $1 + (\dfrac{1}{c})^2 = \text{cosec}^2\theta$

$\therefore$ $\dfrac{1+c^2}{c^2} = \text{cosec}^2\theta$

$\therefore$ $\dfrac{c^2}{c^2+1} = \sin^2\theta \Rightarrow \sin\theta = \sqrt{\dfrac{c^2}{c^2+1}}$ or $\sin\theta = \dfrac{c}{\sqrt{c^2+1}}$

proved .

8, Eliminate θ from the following equation.

$x = r\cos\theta$ and $y = r\sin\theta$

Soln: $x = r\cos\theta$ squarring it, $x^2 = r^2\cos^2\theta$ …………………(i)

Again $y = r\sin\theta$ squarring it, $y^2 = r^2\sin^2\theta$ …………………(ii)

(i) + (ii) $\Rightarrow x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\Rightarrow x^2 + y^2 = r^2$

Ex 9. Eliminate θ from the following equation.

$$x = r\cos\theta\cos\phi \quad , \quad y = r\cos\theta\sin\phi \quad \text{and} \quad z = r\sin\theta$$

Sol^n: given $x = r\cos\theta\cos\phi$

$$\Rightarrow \frac{x}{r} = \cos\theta\cos\phi$$

$$\therefore \left(\frac{x}{r}\right)^2 = (\cos\theta\cos\phi)^2 \quad \Rightarrow \left(\frac{x}{r}\right)^2 = \cos^2\theta\cos^2\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Again given $y = r\cos\theta\sin\phi \Rightarrow \frac{y}{r} = \cos\theta\sin\phi$

$$\therefore \left(\frac{y}{r}\right)^2 = (\cos\theta\sin\phi)^2 \quad \Rightarrow \quad \left(\frac{y}{r}\right)^2 = \cos^2\theta\sin^2\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

(i) + (ii) $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi$

$$= \cos^2\theta(\cos^2\phi + \sin^2\phi)$$

$$= \cos^2\theta$$

$$\therefore \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \cos^2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Again, $z = r\sin\theta \Rightarrow \frac{z}{r} = \sin\theta$

$$\therefore \left(\frac{z}{r}\right)^2 = \sin^2\theta \quad \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

(iii) + (iv) $\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = \cos^2\theta + \sin^2\theta$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = 1$$

$$\Rightarrow x^2 + y^2 + z^2 = r^2 \quad \text{Ans.}$$

Ex 10. If $\sin^4\theta + \sin^2\theta = 1$ then prove that $\tan^4\theta - \tan^2\theta = 1$

Soln: $\sin^4\theta + \sin^2\theta = 1$

$$\Rightarrow (1 - \cos^2\theta)^2 + 1 - \cos^2\theta = 1$$

$$\Rightarrow \left(1 - \frac{1}{\sec^2\theta}\right)^2 + 1 - \frac{1}{\sec^2\theta} = 1$$

$$\Rightarrow \left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right)^2 + 1 - \frac{1}{\sec^2\theta} = 1$$

$$\Rightarrow \frac{\tan^4\theta}{(1+\tan^2\theta)^2} - \frac{1}{1+\tan^2\theta} = 0$$

$$\Rightarrow \frac{\tan^4\theta - 1 - \tan^2\theta}{(1+\tan^2\theta)^2} = 0 \quad \Rightarrow \tan^4\theta - \tan^2\theta = 1 \quad \text{Proved .}$$

H.W

Ex1. Prove that $(\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{1+\cos\theta}{1-\cos\theta}$

Ex 2. Prove that $(\sec\theta + \tan\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$

Ex 3. Prove that $\tan\theta - \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta$

Ex 4. Prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$ - secθ = secθ - $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

Ex 5. Prove that $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Ex 6. If $\cos^2\theta + \cos^4\theta = 1$ then prove that

(i) $\tan^2\theta + \tan^4\theta = 1$

(ii) $\sin\theta + \sin^2\theta = 1$

Ex14. If $\sin\theta + \sin^2\theta = 1$ then prove that $\cos^2\theta - \cos^4\theta = 1$

Ex 7. Eliminate θ from the following

(i) $x = a\sec\theta$, $y = b\tan\theta$

(ii) $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

(iii) $x = a(\cos\theta - \sin\theta)$, $y = b(\sin\theta + \cos\theta)$

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2.2
Trigonometric ratios of some standard Angles .

Here we will determine the trigonometrical ratios(or functions)of some standard angle .These few standard angles are 0^o ,30^o ,45^o , 60^o , 90^o and 180^o .
The standard angles are 0^o ,30^o ,45^o , 60^o , and 90^o and 180^o are tabulated as follows .

θ	0^o	30^o	45^o	60^o	90^o	180^o
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞

Worked out Example .

Ex.1 For $\theta = 45^o$ prove that $\sin^2\theta + \cos^2\theta = 1$

Soln: $\sin^2\theta + \cos^2\theta$

$= \sin^2 45^o + \cos^2 45^o$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Ex.2 If $\theta = 30^o$ prove that $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Soln : L.h.s $\sin 2\theta = \sin 2.30^o$

$= \sin 60^o = \dfrac{\sqrt{3}}{2}$............(i)

R.h.s $\dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2\tan 30^o}{1+\tan^2 30^o}$

$$= \frac{2\frac{1}{\sqrt{3}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{4}{3}} = \frac{\sqrt{3}}{2} \text{............(ii)}$$

From (i) and (ii) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ proved .

Ex.3 Prove that $\dfrac{2\tan 30^o}{1-\tan^2 30^o} = \sqrt{3}$

Soln: $\dfrac{2\tan 30^o}{1-\tan^2 30^o} = \dfrac{2\frac{1}{\sqrt{3}}}{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \dfrac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3}$ proved .

H.W

Ex. 1 For $\theta=45^o$ prove that $1+\tan^2\theta = \sec^2\theta$

Ex.2 Prove that $\sqrt{\dfrac{1+\cos 30^o}{1-\cos 30^o}} = \sec 60^o + \tan 60^o$

Ex.3 Find the value of

(i) $(\sqrt{3}+1)(3-\cot 30^o) + 2\sin 60^o - \tan^2 60^o$ (Ans:0)

(ii) $4\sin^2 45^o + \tan^2 60^o + \text{cosec}^2 30^o$ (Ans: 9)

Ex.5 Prove that $\cos^2 30^o - \sin^2 30^o = \cos 60^o$

Ex.6 Simplify , $(\sqrt{3}+1)(3 - \cot 30^o) + 2\sin 60^o - \tan^3 60^o$ Ans.0

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2.3
Trigonometric ratios of Associated Angles .

2.3.1 Two angles are said to be associated angle if sum or difference is either zero or multiple of 90^o. For example , $-\theta$, $90^o \pm \theta$, $180^o \pm \theta$ etc .

2.3.2 Trigonometric Ratios of $(-\theta)$

Let the line $\overrightarrow{OA}$ makes an angle –θ with $\overrightarrow{OX}$ Then the line $\overrightarrow{OA'}$ makes angle θ with $\overrightarrow{OX}$ as shown in the figure 2.6.Let P is a point on $\overrightarrow{OA}$ and P$'$ is a point on $\overrightarrow{OA'}$ such that OP = OP$'$

The co-ordinate of the point P$'$ is (x ,y) So the co-ordinate of the point P is (x ,-y).

Then , cos(-θ) = x co-ordinate of the point P = x = x co-ordinate of the point P$'$ = cos θ

sin(-θ) = y co-ordinate of the point P = -y = -y co-ordinate of the point P$'$ = -sin θ

∴ cos(-θ) = cosθ and sin(-θ) = - sin θ

Now , $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin\theta}{\cos\theta}$ = - tanθ

$\cot(-\theta) = \dfrac{\cos(-\theta)}{\sin(-\theta)} = \dfrac{\cos\theta}{-\sin\theta}$ = - cotθ

$\sec(-\theta) = \dfrac{1}{\cos(-\theta)} = \dfrac{1}{\cos\theta}$ = secθ

$\text{cosec}(-\theta) = \dfrac{1}{\sin(-\theta)} = \dfrac{1}{-\sin\theta}$ = - cosecθ

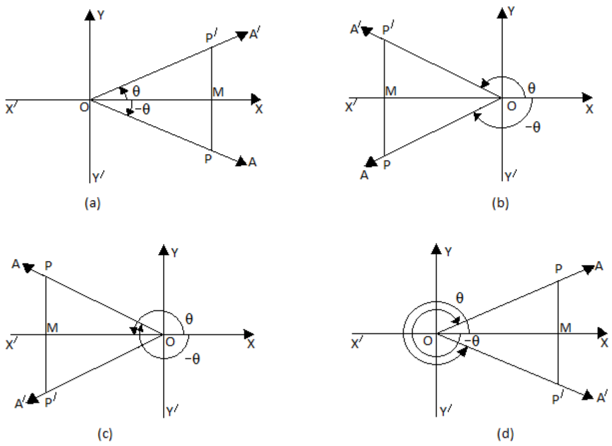


Fig.2.4.1

2.3.3 Signs of Trigonometric Ratios :

(ASTC) rule

A Stands for all
S Stands for sine
T Stands for tangent
C Stands for cosine

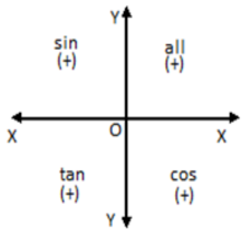
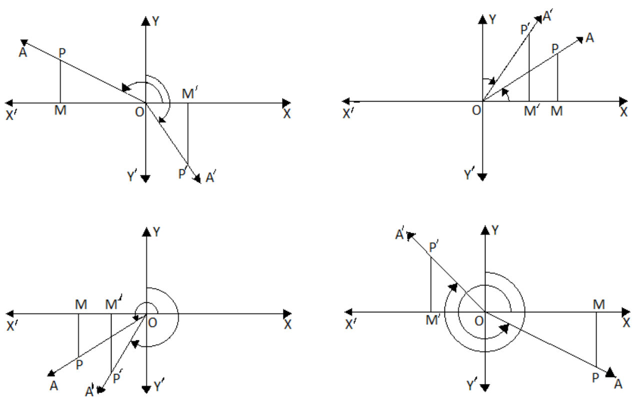


Fig.2.4.2

2.3.4 Trigonometric Ratios of (90^o - θ)

The initial ray $\overrightarrow{OX}$ rotates anticlockwise to the position $\overrightarrow{OA}$ and makes algle θ i.e $\angle AOX = \theta$ .Again it rotating from $\overrightarrow{OA}$ in the same direction and coincide with OY. Then it rotates clockwise to the position $\overrightarrow{OA'}$ and makes angle θ such that $\angle YOA' = \theta$. $\angle XOA' = 90^o - \theta$ .Let P,P$'$ be two points on $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ respectively such that OP = OP$'$. Now, draw ⊥ PM and P$'$M$'$ on OX . ∴ ⊔ OPM = ⊔ OM$'$P$'$.

∴ OM$'$ =MP ,M$'$P$'$ = OM and OP=OP$'$.

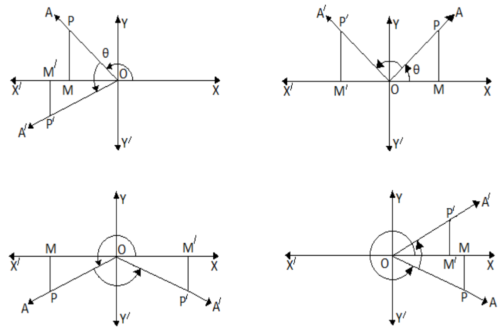


$$\sin(90^o - \theta) = \frac{M^/P^/}{OP^/} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90^o - \theta) = \frac{OM^/}{OP^/} = \frac{MP}{OP} = \sin\theta$$

$$\tan(90^o - \theta) = \frac{\sin(90^o - \theta)}{\cos(90^o - \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

fig 2.4.3



Similarly, $\cot(90^o - \theta) = \tan\theta$;

$$\mathrm{cosec}(90^o - \theta) = \frac{1}{\sin(90o - \theta)} = \frac{1}{\cos\theta} = \sec\theta$$

Similarly, $\sec(90^o - \theta) = \mathrm{cosec}\theta$;

2.3.5 Trigonometric Ratios of $(90^o + \theta)$

The initial ray $\overrightarrow{OX}$ rotates anticlockwise to the position $\overrightarrow{OA}$ and makes algle θ i.e $\angle AOX = \theta$. Again it rotates 90^o from $\overrightarrow{OA}$ in the same direction and gets position $\overrightarrow{OA^/}$. $\angle AOA^/ = 90^o$ and $\angle XOA^/ = 90^o + \theta$.

fig 2.4.4

Let P,P$^/$ be two points on $\overrightarrow{OA}$ and $\overrightarrow{OA^/}$ respectively such that OP = OP$^/$. Now, draw $\perp$ PM and P$^/$M$^/$ on OX . $\therefore \sqcup$ OPM = $\sqcup$ OM$^/$P$^/$.
$\therefore$ OM$^/$ =MP ,M$^/$P$^/$ = OM .

$$\therefore \sin(90^o + \theta) = \frac{M^/P^/}{OP^/} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90^o + \theta) = \frac{OM^/}{OP^/} = \frac{-MP}{OP} = -\sin\theta$$

$$\tan(90^o + \theta) = \frac{\sin(90^o + \theta)}{\cos(90^o + \theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

similarly, $\cot(90^o + \theta) = -\tan\theta$;

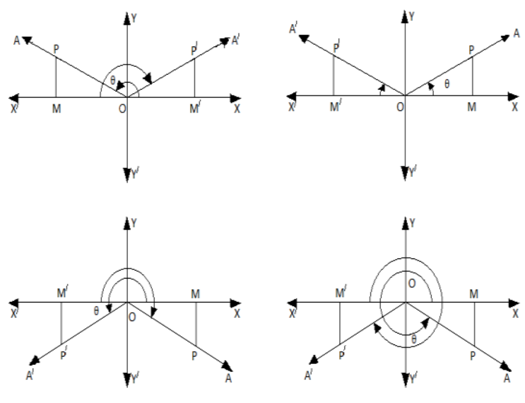
$$\mathrm{cosec}(90^o + \theta) = \frac{1}{\sin(90o + \theta)} = \frac{1}{\cos\theta} = \sec\theta$$

Similarly, $\sec(90^o + \theta) = \mathrm{cosec}\theta$;

2.3.6 Trigonometric Ratios of $(180^o - \theta)$

The initial ray $\overrightarrow{OX}$ rotates anticlockwise to the position $\overrightarrow{OA}$ and makes algle θ i.e $\angle AOX = \theta$. Again it rotating from $\overrightarrow{OA}$ in the same direction and coincide with OX$^/$. Again from OX it rotates clockwise to the position $\overrightarrow{OA^/}$ and makes angle $\angle X^/OA^/ = -\theta$

$\therefore \angle XOA^/ = 180^o - \theta$



Let P,P$^/$ be two points on $\overrightarrow{OA}$ and $\overrightarrow{OA^/}$ respectively such that OP = OP$^/$. Now, draw $\perp$ PM and P$^/$M$^/$ on XOX$^/$. $\therefore \sqcup$ OPM = $\sqcup$ OM$^/$P$^/$.
$\therefore$ OM$^/$ = -OM ,M$^/$P$^/$ =MP and OP=OP$^/$.

$$\sin(180^o - \theta) = \frac{M^/P^/}{OP^/} = \frac{MP}{OP} = \sin\theta$$

$$\cos(180^o - \theta) = \frac{OM^/}{OP^/} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180^o - \theta) = \frac{\sin(180^o - \theta)}{\cos(180^o - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

Similarly,

$\cot(180^o - \theta) = -\cot\theta$ Fig 2.4.5

$\text{cosec}(180^o - \theta) = \dfrac{1}{\sin(180^o - \theta)} = \dfrac{1}{\sin\theta} = \text{cosec}\theta$ Similarly, $\sec(180^o - \theta) = -\sec\theta$;

2.4.7 Trigonometric Ratios of $(180^o + \theta)$

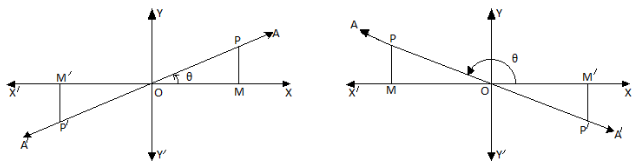
The initial ray $\overrightarrow{OX}$ rotates anticlockwise to the position $\overrightarrow{OA}$ and makes algle θ i.e $\angle AOX = \theta$. Again $\overrightarrow{OA}$ rotates in the same direction and makes angle 180^o with $\overrightarrow{OA^/}$.

$\therefore \angle XOA^/ = 180^o + \theta$

Let P,$P^/$ be two points on $\overrightarrow{OA}$ and $\overrightarrow{OA^/}$ respectively such that OP = $OP^/$. Now, draw perpendicular PM and $P^/M^/$ on $XOX^/$. $\therefore$ ⊔ OPM = ⊔ $OM^/P^/$.

$\therefore$ $OM^/$ = -OM , $M^/P^/$ = -MP and OP=$OP^/$.

fig 2.4.6



$$\sin(180^o + \theta) = \sin XOP^/ = \frac{M^/P^/}{OP^/} = \frac{MP}{OP} = -\sin\theta$$

$$\cos(180^o + \theta) = \cos XOP^/ = \frac{OM^/}{OP^/} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180^o + \theta) = \frac{\sin(180^o + \theta)}{\cos(180^o + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

Similarly, $\cot(180^o + \theta) = \cot\theta$;

$\text{cosec}(180^o + \theta) = \dfrac{1}{\sin(180^o + \theta)} = \dfrac{1}{-\sin\theta} = -\text{cosec}\theta$ Similarly, $\sec(180^o + \theta) = -\sec\theta$;

Important Rule : To find the value of any trigonometric ratios of associated angle

$\varphi = (n \times 90^o \pm \theta)$ where n is an integer .

Case 1. If n is an odd integer ,then the trigonometrical ratios are to be changed as follows .

$\sin(n\times 90^o \pm \theta) = \pm\cos\theta$
$\cos(n\times 90^o \pm \theta) = \pm\sin\theta$
$\tan(n\times 90^o \pm \theta) = \pm\cot\theta$
$\cot(n\times 90^o \pm \theta) = \pm\tan\theta$
$\sec(n\times 90^o \pm \theta) = \pm\text{cosec}\theta$
$\text{cosec}(n\times 90^o \pm \theta) = \pm\sec\theta$

Case II. If n is an even integer ,then the trigonometrical ratios will not be changed as shown below, only the sign + or –ve before the trigonometrical ratios will be changed according to the ASTC rule .

$\sin(n\times 90^o \pm \theta) = \pm\sin\theta$
$\cos(n\times 90^o \pm \theta) = \pm\cos\theta$
$\tan(n\times 90^o \pm \theta) = \pm\tan\theta$
$\cot(n\times 90^o \pm \theta) = \pm\cot\theta$
$\sec(n\times 90^o \pm \theta) = \pm\sec\theta$
$\text{cosec}(n\times 90^o \pm \theta) = \pm\text{cosec}\theta$

Worked out example:

Ex 1. Find (i) $\sin 135^o$ (ii) $\cos 150^o$ (iii) $\sin(-240^o)$ (iv) $\sin 315^o$

Soln (i) : $\sin 135^o$

$= \sin(180^o - 45^o)$

$= \sin 45^o$

$= \frac{1}{\sqrt{2}}$ Ans.

Soln (ii) : $\sin 150^o$

$= \sin(180^o - 30^o)$

$= \sin 30^o$

$= \frac{1}{2}$ Ans

Soln (iii) : $\sin(-240^o)$

$= -\sin 240^o$

$= -\sin(180^o + 60^o)$

$= -\sin 60^o = -\frac{\sqrt{3}}{2}$ Ans.

Soln (iv) : $\sin 315^o$

$= \sin(4\times 90^o - 45^o)$

$= -\sin 45^o$

$= -\frac{1}{\sqrt{2}}$ Ans.

Ex 2. Find $\sin(-585^o)$

Soln: $\sin(-585^o) = -\sin(585^o)$

$= -\sin(6\times 90^o + 45^o)$

$= -(-\sin 45^o) = \sin 45^o = \frac{1}{\sqrt{2}}$ Ans.

Ex3. Find the value of $\operatorname{Sin}495^o$

Soln : $\operatorname{Sin}495^o$

$= \sin(5\times 90^o + 45^o)$

$= -\cos 45^o = -\frac{1}{\sqrt{2}}$

Ex6. prove that $\cos^2\frac{\pi}{4} + \cos^2\frac{3\pi}{4} + \cos^2\frac{5\pi}{4} + \cos^2\frac{7\pi}{4} = 2$

Soln: R.H.S. $\cos^2\frac{\pi}{4} + \cos^2\frac{3\pi}{4} + \cos^2\frac{5\pi}{4} + \cos^2\frac{7\pi}{4}$

$=(\frac{1}{\sqrt{2}})^2 + (\cos\frac{3\pi}{4})^2 + (\cos\frac{5\pi}{4})^2 + (\cos\frac{7\pi}{4})^2$

$=\frac{1}{2} + \cos(\pi - \frac{\pi}{4})^2 + \cos(\pi + \frac{\pi}{4})^2 + \cos\{(2\pi - \frac{\pi}{4})^2\}$

$=\frac{1}{2} + (-\cos\frac{\pi}{4})^2 + (\cos\frac{\pi}{4})^2 + (\cos\frac{\pi}{4})^2$

$=\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$ proved.

Ex8. prove that $\frac{2\tan 30^o}{1-\tan^2 30^o} = \sqrt{3}$

Sol^n : $\frac{2\tan 30^o}{1-\tan^2 30^o} = \frac{2\frac{1}{\sqrt{3}}}{1-(\frac{1}{\sqrt{3}})^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3}$ proved .

Ex9. prove that $\sin(90^o+\theta)\sin(270^o+\theta) + \sin(270^o+\theta)\cos(180^o+\theta) = 0$

Soln: L.h.s $\sin(90^o+\theta)\sin(270^o+\theta) + \sin(270^o+\theta)\cos(180^o+\theta)$

$= \cos\theta(-\cos\theta) - \cos\theta(-\cos\theta)$

$= -\cos^2\theta + \cos^2\theta = 0$ proved.

H.W

Ex 1. Find the value of
(i) $\tan 4620^o$ (ii) $\cos(-1575^o)$ (iii) $\sin 870^o$ (iv) $\sin(-1580^o)$ (v) $\tan 1035^o$
(vi) $\sin(-960^o)$ (vii) $\cos(-480^o)$ (viii) $\sec 585^o$

Ex 2. Prove that (i) $\sec^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4} = 2$

Ex 4. Prove that $\cos 470^o \sin 510^o - \sin 690^o \cos 390^o = 0$

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2.4
Trigonometric ratios of compound angles

2.4.1 Compound angles : The angles which are formed by addition and subtraction of two or more angles are known as compound angles .
Eg. For the angles , A , B , C , the angles A+B , A-B , B+C , B-C are compound angles .

2.4.2 Addition formula for compound angles :

Addition formula of sine and cosine for compound angles (A+B) are

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$
(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Note : (i) $\sin(A \pm B) \neq \sin A \pm \sin B$

(ii) $\cos(A \pm B) \neq \cos A \pm \cos B$

2.4.3 Subtraction formulae for compound angles :

Subtraction formula of sine and cosine for compound angles (A - B) are

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
(ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Note: (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (ii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

proof of (i) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A\cos B+\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\therefore \quad \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

proof of (ii) $\quad \tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)}$

$$= \frac{\sin A\cos B-\sin B\cos A}{\cos A\cos B+\sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B-\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B+\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\therefore \quad \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B} \qquad \text{proved.}$$

Cor :
(i) sin(A+B+C) = sinAcosBcinC+ cosAsinBcosC + cosAcosBsinC -sinAsinBsinC

(ii) cos(A+B+C) = cosAcosBsinC - sinAsinBcosC + sinAcosBsinC - cosAsinBsinC

Cor : (iii) $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$

Worked out example :

Ex .1 Find the value of $\sin 75^o$

Soln: $\sin 75^o = \sin(45^o+30^o)$

$= \sin 45^o \cos 30^o + \cos 45^o \sin 30^o$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} \qquad \text{Ans.}$$

Ex .2 Find the value of $\tan 75^o$

Soln: $\tan 75^o = \tan(30^o + 45^o)$

$$= \frac{\tan 30^o + \tan 45^o}{1 - \tan 30^o \tan 45^o}$$

$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad \text{Ans .}$$

Ex 3. prove that $\tan(45^O - \theta) = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

Proof : $\tan(45^O - \theta) = \dfrac{\tan 45^o - \tan\theta}{1 + \tan 45^o \tan\theta}$

$$= \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$= \frac{1 - \frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\cos\theta - \sin\theta}{\cos\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \quad \text{proved.}$$

Ex .6 Prove that $\tan 26^o + \tan 19^o + \tan 26^o \tan 19^o = 1$

Proof : $\tan 45^o = \tan(26^o + 19^o)$

$$\Rightarrow 1 = \frac{\tan 26^o + \tan 19^o}{1 - \tan 26^o \tan 19^o}$$

$$\Rightarrow 1 - \tan 26^o \tan 19^o = \tan 26^o + \tan 19^o$$

$$\Rightarrow 1 = \tan 26^o + \tan 19^o + \tan 26^o \tan 19^o \quad \text{proved .}$$

Ex .7 Prove that $\tan(45^O - \theta)\tan(45^O + \theta) = 1$

Proof : L.h.s $= \tan(45^O - \theta)\tan(45^O + \theta)$

$$= \frac{\tan 45^o - \tan\theta^o}{1 + \tan 45^o \tan\theta^o} \cdot \frac{\tan 45^o + \tan\theta^o}{1 - \tan 45^o \tan\theta^o}$$

$$= \frac{1 - \tan\theta^o}{1 + \tan\theta^o} \cdot \frac{1 + \tan\theta^o}{1 - \tan\theta^o} = 1 \quad \text{proved.}$$

Ex. 8 If $\alpha + \beta = \dfrac{\pi}{4}$ prove that $(1 + \tan\alpha)(1 + \tan\beta) = 2$

Proof : Given , $\alpha + \beta = \dfrac{\pi}{4}$

Now , $\tan(\alpha + \beta) = \tan\dfrac{\pi}{4}$

$$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$$

$\Rightarrow \tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$

$\Rightarrow \tan\alpha + \tan\beta + \tan\alpha\tan\beta = 1$

Now , R.H.S. $(1+\tan\alpha)(1+\tan\beta)$

$\Rightarrow 1 + \tan\alpha + \tan\beta + \tan\alpha\tan\beta$

$\Rightarrow 1 + 1$ $\because \tan\alpha + \tan\beta + \tan\alpha\tan\beta = 1$

$= 2$ proved .

H.W

Ex.1 Find the value

(i) sec(-75^o) (ii) cot (-105^o) (iii)cos15^o (iv) cos(105^o) (v) cot (165^o)
(vi) sec 255^o (vii) cos (285^o) (viii) cos(195^o)

Ex.2 Prove that $\sin A + \cos A = \sqrt{2}\cos(45^o - A)$

Ex .3 (i) Prove that $\tan(45^O - \theta) = \dfrac{1-\tan\theta}{1+\tan\theta}$

(ii) Prove that $\tan(45^O + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$

Ex .4 Prove that $\cos A - \sin A = \sqrt{2}\sin(45^o - A) = \sqrt{2}\cos(45^o + A)$

Ex. 5 Prove that $\cos 65^o + \sin 65^o = \sqrt{2}\cos 20^o$

Ex .6 Prove that $\sin 105^o + \cos 105^o = \cos 45^o$

Ex.7 Prove that (i) $\dfrac{\cos 10^0 + \sin 10^0}{\cos 10^0 - \sin 10^0} = \tan 55^o$ (ii) $\dfrac{\cos 18^0 - \sin 18^0}{\cos 18^0 + \sin 18^0} = \tan 27^o$

Ex.8 If $A+B+C=\dfrac{\pi}{2}$ then prove that $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$

Ex .9 If $A+B+C = \pi$ and $\cos A = \cos B\cos C$ then prove that

(a) $\tan A = \tan B + \tan C$

(b) $\cot B\cot C = \dfrac{1}{2}$

Ex.10 Prove that $\dfrac{\sin(B-C)}{\cos B\cos C} = \tan B - \tan C$

Ex. 11 Prove that $\dfrac{\sin(B-C)}{\cos B\cos C} = \dfrac{\sin(C-A)}{\cos C\cos A} = \dfrac{\sin(A-B)}{\cos A\cos B} = 0$

Ex .12. Prove that

(i) $\tan 16^o + \tan 29^o + \tan 16^o\tan 29^o = 1$

(ii) $\tan 26^o + \tan 19^o + \tan 26^o\tan 19^o = 1$

(ii) $(1- \tan 55^o) \tan 100^o = 1+ \tan 55^o$

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2.5
Transformation of Sums and Products

From the above chepter we have ,

sin(A+B) + sin(A+B) = 2 sinA cosB ……………………………….(v)

sin(A+B) - sin(A+B) = 2 cosA sinB ………………………………..(vi)

cos(A+ B) + cos(A- B) = 2 cosA cosB ……………………………..(vii)

cos(A-B) - cos(A + B) = 2 sinA sinB ……………………………..(viii)

2.5.1 Expression of sum and difference as product:

Let A + B = C and A - B = D $\therefore$ A = $\frac{C+D}{2}$ and B = $\frac{C-D}{2}$

Putting the value of A and B in (v)

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \quad ….(ix)$$

Putting the value of A and B in (vi) ,

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \quad ….(x)$$

Putting the value of A and B in (vii) , $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$ …. (xi)

Putting the value of A and B in (viii) ,

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \quad ….(xii)$$

Remember:

(ix) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(x) $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

(xi) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(xii) $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

Worked out examples:

Ex.1 Express as sum (i) $2\cos 2\theta \sin 5\theta$

(i) Soln: $2\cos 2\theta \sin 5\theta$

$= \sin (2\theta + 5\theta) - \sin (2\theta - 5\theta)$

$= \sin 7\theta - \sin (-3\theta)$

$= \sin 7\theta + \sin 3\theta$ Ans.

Ex.2 Prove that $\sin 25^o + \cos 25^o = \sqrt{2}\cos 20^o$

Proof: L.h.s $= \sin 25^o + \cos 25^o = \sin(90^o - 65^o) + \cos 25^o$

$= \cos 65^o + \cos 25^o$ [from (xi)]

$= 2\cos\frac{65^o + 25^o}{2}\cos\frac{65^o - 25^o}{2}$

$= 2\cos\frac{90^o}{2}\cos\frac{40^o}{2}$

$= 2\cos 45^o \cos 20^o$

$= 2\frac{1}{\sqrt{2}}\cos 20^o$

$= \sqrt{2}\sqrt{2}\frac{1}{\sqrt{2}}\cos 20^o = \sqrt{2}\cos 20^o$ proved.

Ex .3 Find the value of $\sin 75^o \sin 15^o$

Soln: $\sin 75^o \sin 15^o$

$= \frac{1}{2}[2\sin 75^o \sin 15^o]$

$= \frac{1}{2}[\cos(75^o - 15^o) - \cos(75^o + 15^o)]$

$= \frac{1}{2}[\cos(60^o) - \cos(90^o)]$

$= \frac{1}{2}[\frac{1}{2} - 0] = \frac{1}{4}$ Ans.

Ex .4 Prove that $\cos 20^o + \cos 100^o + \cos 140^o = 0$

Proof: L.h.s $= \cos 20^o + \cos 100^o + \cos 140^o$

$= \cos\frac{120^0}{2}\cos\frac{20^0 - 100^0}{2} + \cos 140^o$

$= 2\cos 60^o \cos(-40^o) + \cos 140^o$

$= 2\frac{1}{2}\cos 40^o + \cos 140^o$

$= \cos 40^o + \cos 140^o$

$= 2\cos\frac{180^0}{2}\cos\frac{40^0 - 140^0}{2}$

$= 2.\cos 90^o \cos(-50^o)$

$= 2.\ 0.\cos(-50^o) = 0$ = R.H.S proved

Ex .5 Express 4cosAcosBcosC as the sum of four cosines

Soln: $4\cos A\cos B\cos C$

$= 2\cos A\,[\,2\cos B\cos C\,]$

$= 2\cos A\,[\cos(B + C) + \cos(B - C)]$

$= 2\cos A\cos(B + C) + 2\cos A\cos(B - C)$

$= \cos\{A + (B + C)\} + \cos\{A - (B + C)\} + \cos\{A + (B - C)\} + \cos\{A - (B - C)\}$

$= \cos\{A + B + C\} + \cos\{A - B - C\} + \cos\{A + B - C\} + \cos\{A - B + C\}$ Ans.

Ex .6 Show that $\cos 20^o \cos 40^o \cos 60^o \cos 80^o = \frac{1}{16}$

Proof : L.h.s = $\cos 20^o \cos 40^o \cos 60^o \cos 80^o$

$$= \frac{1}{2} \cos 20^o \cos 40^o \cos 80^o$$

$$= \frac{1}{4} (2\cos 20^o \cos 40^o)\cos 80^o$$

$$= \frac{1}{4} (\cos 60^o + \cos 20^o)\cos 80^o$$

$$= \frac{1}{4} \cos 60^o \cos 80^o + \frac{1}{4} \cos 20^o \cos 80^o$$

$$= \frac{1}{4} . \frac{1}{2} \cos 80^o + \frac{1}{4} \frac{1}{2} (2 \cos 20^o \cos 80^o)$$

$$= \frac{1}{8} \cos 80^o + \frac{1}{8} (\cos 100^o + \cos 60^o)$$

$$= \frac{1}{8} \cos 80^o + \frac{1}{8} (\cos 100^o + \frac{1}{2})$$

$$= \frac{1}{8} \cos 80^o + \frac{1}{8} \cos 100^o + \frac{1}{8} . \frac{1}{2}$$

$$= \frac{1}{8} \cos 80^o + \frac{1}{8} \cos (180^o - 80^o) + \frac{1}{16}$$

$$= \frac{1}{8} \cos 80^o - \frac{1}{8} \cos 80^o + \frac{1}{16} = \frac{1}{16} = \text{R.H.S.} \quad \text{proved .}$$

2.5

Exercise.

Ex.1 Express as sum

(i) $2\sin 6\theta \sin 8\theta$ (ii) $2\cos 5\theta \cos 3\theta$

Ans: (i) $(\cos 2\theta - \cos 14\theta)$; (ii) $(\cos 8\theta + \cos 2\theta)$

Ex .2 Express as product .

(i) $(\sin 2\theta - \sin 8\theta)$ (ii) $(\cos 12\theta - \cos 8\theta)$

Ans. (i) $-2\cos 5\theta \sin 3\theta$; (ii) $-2\sin 10\theta \sin 2\theta$

Ex .3 Prove that

(i) $\sin 20^o \sin 40^o \sin 60^o \sin 80^o = \frac{3}{16}$

(ii) $\sin 10^o \sin 30^o \sin 50^o \sin 70^o = \frac{1}{16}$

Ex .4 Prove that $\cos 20^o + \cos 100^o + \cos 140^o = 0$

Ex . 5 Prove that $\sin 20^o + \sin 140^o - \cos 10^o = 0$

Ex . 6 Show that $\cos 20^o \cos 40^o \cos 80^o = \frac{1}{8}$

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2.6
Multiple and submultiple Angle

We have in previous chapter that

Addition formula of sine and cosine for compound angles (A+B) are

(i) sin(A+B) = sinA cosB + cosA sinB

(ii) cos(A+B) = cosA cosB - sinA sinB

Putting A= B in (i) we have

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$
$$\Rightarrow \text{Sin}2A = 2\sin A \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Again putting A = B in (ii) we have

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$
$$\Rightarrow \text{Cos } 2A = \cos^2 A - \sin^2 A \ \ldots\ldots\ldots..(2)$$
$$\Rightarrow \text{Cos } 2A = \cos^2 A - (1- \cos^2 A)$$
$$= 2\cos^2 A - 1 \ \ldots\ldots\ldots\ldots\ldots(3)$$
$$= 2(1- \sin^2 A)-1$$
$$= 1- 2\sin^2 A \ \ldots\ldots\ldots\ldots..(4)$$

From (3) $\text{Cos } 2A = 2\cos^2 A - 1$

$$\therefore \text{Cos } 2A + 1 = 2\cos^2 A \ \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

again from (4)

$$\text{Cos}2A= 1- 2\sin^2 A$$
$$\therefore 2\sin^2 A = 1- \cos 2A \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\frac{(6)}{(5)} \Rightarrow \frac{2\sin 2A}{2\cos 2A} = \frac{1- \cos 2A}{1+ \cos 2A}$$

$$\Rightarrow \frac{\sin 2A}{\cos 2A} = \frac{1- \cos 2A}{1+ \cos 2A}$$

$$\tan 2A = \frac{1- \cos 2A}{1+ \cos 2A} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Also we have from previous chapter that (iii) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Again putting A = B in (iii) we have

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Again putting A = B in $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$

$$\cot(A+A) = \frac{\cot A \cot A - 1}{\cot A + \cot A}$$

$$\Rightarrow \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \text{ ....(9)}$$

Note:1. $1+\sin 2A = 1+2\sin A\cos A$

$= \sin^2 A + \cos^2 A + 2\sin A\cos A$

$$\mathbf{1+\sin 2A = (\sin A + \cos A)^2}$$

Note: 2. $1-\sin 2A = 1-2\sin A\cos A$

$= \sin^2 A + \cos^2 A - 2\sin A\cos A$

$$\mathbf{1-\sin 2A = (\sin A - \cos A)^2}$$

Note: 3. $\sin 2A = 2\sin A\cos A$

$$= \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$$

$$= \frac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\sin^2 A + \cos^2 A}{\cos^2 A}}$$

$$= \frac{\frac{2\sin A}{\cos A}}{\frac{\sin^2 A}{\cos^2 A} + 1} = \frac{2\tan A}{1+\tan^2 A}$$

$$\therefore \sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

Note: 4. $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}$$

$$= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\therefore \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

Trigonometric ratio of 3A

(a) $\sin 3A = 3\sin A - 4\sin^3 A$

(b) $\cos 3A = 4\cos^3 A - 3\cos A$

(c) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof: (a) $\sin 3A = \sin(2A+A)$

$= \sin A\cos 2A + \cos A\sin 2A$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A\,(1 - 2\sin^2 A) + \cos A\, 2\sin A \cos A$

$= \sin A - 2\sin^3 A + 2\sin A \cos^2 A$

$= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

$\therefore\ \sin 3A = 3\sin A - 4\sin^3 A$

Proof: (b) $\cos 3A = \cos(2A + A)$

$= \cos A \cos 2A - \sin A \sin 2A$

$= \cos A(2\cos^2 A - 1) - \sin A\, 2\sin A \cos A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$\therefore\ \cos 3A = 4\cos^3 A - 3\cos A$

Proof: (c) $\tan 3A = \tan(2A + A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1 - \tan^2 A}}{1 - \tan A \dfrac{2\tan A}{1 - \tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1 - \tan^2 A}}{1 - \dfrac{2\tan^2 A}{1 - \tan^2 A}}$$

$$= \frac{\dfrac{3\tan A - \tan^3 A}{1 - \tan^2 A}}{\dfrac{1 - \tan^2 A - 2\tan^2 A}{1 - \tan^2 A}} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\therefore\ \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Corollary : (i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin 2\frac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

Worked out example :

Ex.1 If $\cos\theta = \frac{3}{5}$ then find the value of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$

Soln: We know $\sin 2\theta = 2\sin\theta\cos\theta$

Given $\cos\theta = \frac{3}{5}$, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin^2\theta + \left(\frac{3}{5}\right)^2 = 1 \quad \Rightarrow \sin^2\theta = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$$

$$\Rightarrow \sin\theta = \frac{4}{5}$$

Now $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\frac{4}{5}\frac{3}{5} = \frac{24}{25}$,

Again $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{24}{25}}{\frac{-7}{25}} = -\frac{24}{7} \quad \text{Ans.}$$

Ex.2 Prove that $\cot A = \dfrac{\sin 2A}{1-\cos 2A}$

Proof : R.H.S = $\dfrac{\sin 2A}{1-\cos 2A} = \dfrac{2\sin A\cos A}{1-(1-2\sin^2 A)} = \dfrac{2\sin A\cos A}{2\sin^2 A} = \dfrac{\cos A}{\sin A}$ = cot A=L.H.S

Proved.

Ex.3 Prove that $\tan(\frac{\pi}{4}+\theta) = \dfrac{1+\sin 2\theta}{\cos 2\theta}$

Proof : R.H.S $= \dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$

$$= \frac{(\sin\theta+\cos\theta)^2}{\cos^2\theta-\sin^2\theta}$$

$$= \frac{(\sin\theta+\cos\theta)}{\cos\theta-\sin\theta} \quad (\text{ Divide by } \cos\theta)$$

$$= \frac{1+\tan\theta}{1-\tan\theta}$$

$$= \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\theta\tan\frac{\pi}{4}} = \tan(\frac{\pi}{4}+\theta) \text{ proved.}$$

Ex.4 Prove that $\tan(\frac{\pi}{4}+\frac{\theta}{2}) = \sec\theta + \tan\theta$

Proof : $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \dfrac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$= \frac{1+\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1-\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\dfrac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\dfrac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$$

$$= \frac{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}$$

$$= \frac{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})^2}{(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})} = \frac{(1+2\cos\frac{\theta}{2}\sin\frac{\theta}{2})}{c\,os\,\theta} = \frac{(1+\sin\theta)}{c\,os\,\theta} = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta} = \sec\theta+\tan\theta$$

proved.

Ex 6. If $\tan\theta = \dfrac{b}{a}$ prove that $a\cos 2\theta + b\sin 2\theta = a$

Proof : $a\cos 2\theta + b\sin 2\theta = a\,\dfrac{1-\tan^2\theta}{1+\tan^2\theta} + b\,\dfrac{2\tan\theta}{1+\tan^2\theta}$

$$= a\,\frac{1-(\frac{b}{a})^2}{1+(\frac{b}{a})^2} + b\,\frac{2\frac{b}{a}}{1+(\frac{b}{a})^2}$$

$$= a\,.\frac{a^2-b^2}{a^2+b^2} + b.\ \frac{2ab}{a^2+b^2}$$

$$= \frac{a^3 - a\ b^2 + 2ab^2}{a^2 + b^2}$$

$$= \frac{a^3 + ab^2}{a^2 + b^2}$$

$$= \frac{a(a^2 + b^2)}{a^2 + b^2} = a \quad \text{proved.}$$

Ex 7. Express cos4A in terms of cosA .

Soln: $\cos 4A = \cos 2\,(2A) = \cos 2\theta$ $\qquad (2A) = \theta$

$= 2\cos^2\theta - 1$

$= 2\cos^2 2A - 1$

$= 2(\cos 2A)^2 - 1$

$= 2(2\cos^2 A - 1)^2 - 1$

$= 2(4\cos^4 A - 4\cos^2 A + 1) - 1$

$= 8\cos^4 A - 8\cos^2 A + 2 - 1$

$= 8\cos^4 A - 8\cos^2 A + 1$ Ans.

Ex 8. Find the value of $\tan 7\frac{1}{2}^o$

Proof : $$\tan 7\frac{1}{2}^o = \frac{\sin 7\frac{1}{2}^o}{\cos 7\frac{1}{2}^o}$$

$$= \frac{2\sin 7\frac{1}{2}^o \cos 7\frac{1}{2}^o}{2\cos 7\frac{1}{2}^o \cos 7\frac{1}{2}^o}$$

$$= \frac{\sin 15^o}{1 + \cos 15^o}$$

$$= \frac{\sin(45^o - 30^o)}{1 + \cos(45^o - 30^o)}$$

$$= \frac{\sin 45^o \cos 30^o - \sin 30^o \cos 45^o}{1 + \cos 45^o \cos 30^o + \sin 45^o \sin 30^o}$$

$$= \frac{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}}{1 + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} + 2\sqrt{2}} \qquad \text{Ans.}$$

Ex 9. Prove that $\sin 22\frac{1}{2}^o = \frac{1}{2}\sqrt{2 - \sqrt{2}}$

Proof: $\theta = 22\frac{1}{2}^o = \frac{45^o}{2}$

$\Rightarrow 2\theta = 45^o$

Now , $\sin 22\frac{1}{2}^o = \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\cos 45^o}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ proved.

2.6.1 Trigonometric ratios of some important angles:

Trigonometric ratios of angle 18^o , 36^o 54^o , 72^o

Let $\theta = 18^o$

$\Rightarrow 5\theta = 90^o$

$\Rightarrow 2\theta + 3\theta = 90^o$

$\Rightarrow 2\theta = 90^o - 3\theta$

$\Rightarrow \operatorname{Sin} 2\theta = \sin(90^o - 3\theta)$

$\Rightarrow \operatorname{Sin} 2\theta = \cos 3\theta$

$\Rightarrow 2\operatorname{Sin}\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow 2\operatorname{Sin}\theta\cos\theta = \cos\theta(4\cos^2\theta - 3)$ $(\because \cos\theta \neq 0)$

$\Rightarrow 2\operatorname{Sin}\theta = 4\cos^2\theta - 3 \Rightarrow 4\cos^2\theta - 3 - 2\sin\theta = 0$

$\Rightarrow 4(1 - \sin^2\theta) - 3 - 2\sin\theta = 0$

$\Rightarrow 4 - 4\sin^2\theta - 3 - 2\sin\theta = 0$

$\Rightarrow 1 - 4\sin^2\theta - 2\sin\theta = 0$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$

As θ (=18^o) actute angle $\sin\theta$ is positive , so **$\sin 18^o$ =** $\frac{\sqrt{5}-1}{4}$

$\sin^2 18^o + \cos^2 18^o = 1$

$\Rightarrow \cos^2 18^o = 1 - \sin^2 18^o$

$= 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$

$= 1 - \frac{6-2\sqrt{5}}{16}$

$= \frac{10+2\sqrt{5}}{16}$

$\Rightarrow \cos^2 18^o = \dfrac{10+2\sqrt{5}}{16}$

$\Rightarrow$ **$\cos 18^o$ =** $\dfrac{\sqrt{10+2\sqrt{5}}}{4}$ **,** $\cos 18^o > 0$ so it is of positive sign .

$\cos 36^o = 1 - 2\sin^2 18^o$

$= 1 - 2\left(\dfrac{\sqrt{5}-1}{4}\right)^2$

$= 1 - \dfrac{6-2\sqrt{5}}{8} = \dfrac{\sqrt{5}+1}{4}$ $\quad\therefore$ **$\cos 36^o$ =** $\dfrac{\sqrt{5}+1}{4}$

Now , $\sin^2 36^o + \cos^2 36^o = 1$

$\sin 36^o = \sqrt{1 - \cos^2 36^o}$

$= \sqrt{1 - \dfrac{1}{16}\left(6+2\sqrt{5}\right)}$

$= \sqrt{\dfrac{1}{16}\left(10 - 2\sqrt{5}\right)}$

$= \dfrac{1}{4}\sqrt{\left(10 - 2\sqrt{5}\right)}$ $\quad\therefore$ **$\sin 36^o$ =** $\dfrac{1}{4}\sqrt{\left(10 - 2\sqrt{5}\right)}$

By similar process we have ,

$\sin 54^o = \sin(90^o - 36^o) = \cos 36^o = \dfrac{\sqrt{5}+1}{4}$ $\quad\therefore$ **$\sin 54^o$ =** $\dfrac{\sqrt{5}+1}{4}$

$\cos 54^o = \cos(90^o - 36^o) = \sin 36^o = \dfrac{1}{4}\sqrt{\left(10 - 2\sqrt{5}\right)}$ $\quad\therefore$ **$\cos 54^o$ =** $\dfrac{1}{4}\sqrt{\left(10 - 2\sqrt{5}\right)}$

Again , **$\sin 72^o$** $= \sin(90^o - 18^o) = \cos 18^o = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

$\cos 72^o$ $= \cos(90^o - 18^o) = \sin 18^o = \dfrac{\sqrt{5}-1}{4}$

HW

1. If $\tan\theta = t$ then prove that $\sin 2\theta = \dfrac{2t}{1+t^2}$ and $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$
2. Find the value of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$

6. Prove that

(iii) $\cos^4\theta - \sin^4\theta = \cos 2\theta$ (iv) $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

(vii) $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$

(viii) $\dfrac{\sin 2A}{1+\cos 2A}$ = tanA

(xi) $\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ = secx + tanx

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2.7
Trigonometric Identities

If A+B+C = π then B+C = π - A (π = 180°)

$\therefore$ sin(B+C) = sin (π-A) = sinA

$\therefore$ cos (B+C) = cos (π-A) = - cosA

$\therefore$ tan(B+C) = tan (π-A) = - tanA

Similar result can be obtained for A+ C = π - B and A+ B = π - C .

Again A+B+C = π then B+C = π - A

$\therefore \dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2} = \dfrac{\pi}{2}$

Worked out example :

Ex.1 If A+B+C = π then tanA+tanB+tanC = tanA tanB tanC

Proof: Given A+B+C = π

$\Rightarrow$ A+B = π - C

$\Rightarrow$ tan (A+B) = tan (π - C)

$\Rightarrow \dfrac{\tan A+\tan B}{1-\tan A\tan B}$ = - tanC

$\Rightarrow$ $\tan A+\tan B$ = -tanC$(1-\tan A\tan B)$

$\Rightarrow$ $\tan A+\tan B$ = - tanC + $\tan A\tan B$tanC

$\Rightarrow$ $\tan A+\tan B$ + tanC = $\tan A\tan B$tanC proved.

Ex.2 If A+B+C = π then cotAcotB + cotBcotC + cotAcotC = 1

Soln: Given , A+B+C = π

$\Rightarrow$ A+B = π - C

$\Rightarrow$ tan (A+B) = tan (π - C)

$\Rightarrow \dfrac{\tan A+\tan B}{1-\tan A\tan B}$ = - tanC

$\Rightarrow$ $\tan A+\tan B$ = -tanC$(1-\tan A\tan B)$

$\Rightarrow$ $\tan A+\tan B$ = - tanC + $\tan A\tan B$tanC

$\Rightarrow$ $\tan A+\tan B$ + tanC = $\tan A\tan B$tanC

Multiply both side by cotA cotB cotC we have ,

cotA cotB cotC ($\tan A+\tan B$ + tanC) = cotA cotB cotC($\tan A\tan B$tanC)

$\Rightarrow$ cotAcotB + cotBcotC + cotAcotC = 1 proved.

Ex .3 If A+B+C = π then sin2A - sin2B + sin2C = 4cosAsinBcosC

Proof: L.H.S. sin2A - sin2B + sin2C

= (sin2A - sin2B) + sin2C

= 2cos(A+B)sin(A – B)+ 2sinC cosC

= -2cosC sin(A – B) + 2sinC cosC

= 2cosC[sinC - sin(A – B)]

= 2cosC[sin(A+B) - sin(A – B)]

= 2cosC[2cosA sinB]

A+B = π - C

$\therefore$ cos(A+B) =cos(π-C)

= - cosC

$\therefore$ sin(A+B) = sin(π -C)=sinC

$= 4\cos A\sin B\cos C$ proved.

Ex .4 If $A+B+C=\pi$ then $\cos 2A + \cos 2B + \cos 2C = -4\cos A\cos B\cos C - 1$

Proof: L.H.S. $\cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + (2\cos^2 C - 1)$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$ $[\because \cos(A+B) = \cos(\pi - C) = -\cos C]$

$= -2\cos C[\cos(A-B) + \cos(A+B)] - 1$ $[\because -\cos C = \cos(A+B)]$

$= -2\cos C\, 2\cos A\cos B - 1$

$= -4\cos C\cos A\cos B - 1$ proved.

Ex. 5 If $A+B+C=\pi$ then $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Proof : L.H.S. $\sin A + \sin B + \sin C$

$= 2\sin\frac{(A+B)}{2}\cos\frac{(A-B)}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\cos\frac{(A-B)}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{(A-B)}{2} + \sin\frac{C}{2}\right]$ $\left(\because \sin\frac{(A+B)}{2} = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}\right)$

$= 2\cos\frac{C}{2}\left[\cos\frac{(A-B)}{2} + \cos\frac{(A+B)}{2}\right]$ $\left(\because \cos\frac{(A+B)}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\frac{C}{2}\right)$

$= 2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ proved.

Ex. 6 If $A+B+C=\pi$ then $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Proof: L.H.S $\cos A + \cos B + \cos C$

$= 2\cos\frac{(A+B)}{2}\cos\frac{(A-B)}{2} + \cos C$

$= 2\sin\frac{C}{2}\cos\frac{(A-B)}{2} + 1 - 2\sin^2\frac{C}{2}$

$= 2\sin\frac{C}{2}\left[\cos\frac{(A-B)}{2} - 2\sin\frac{C}{2}\right] + 1$

$= 2\sin\frac{C}{2}\left[\cos\frac{(A-B)}{2} - \cos\frac{(A+B)}{2}\right] + 1$

$= 1 + 2\sin\frac{C}{2}\, 2\sin\frac{A}{2}\sin\frac{B}{2}$

$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ proved.

Ex. 7 If $A+B+C=\frac{\pi}{2}$ then $\cot A + \cot B + \cot C = \cot A\cot B\cot C$

Proof: Given , $A+B+C=\frac{\pi}{2}$

$\therefore\ A+B = \frac{\pi}{2} - C$

$\therefore \cot(A+B) = \cot\left(\frac{\pi}{2} - C\right)$

$\Rightarrow \frac{\cot A\cot B - 1}{\cot A + \cot B} = \tan C$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1}{\cot C}$$

$\Rightarrow$ cotC($\cot A \cot B - 1$) = cotA + cotB

$\Rightarrow$ cotCcotAcotB - cotC = cotA + cotB

$\Rightarrow$ cotCcotAcotB = cotA + cotB + cotC proved.

Exercise

Ex.1 If A+B+C = π then

(i) cos2A + cos2B - cos2C = 1 - 4cosAcosBcosC

(ii) cosA + cosB - cosC = -1+ 4cos $\frac{A}{2}$ cos $\frac{B}{2}$ cos $\frac{C}{2}$

(iii) sin A + sin B - sinC = 4sin $\frac{A}{2}$ sin $\frac{B}{2}$ cos $\frac{C}{2}$

Ex. 2 If A+B+C = π then

(i) cotA cotB + cotB cotC + cotA cotC = 1

(ii) cot $\frac{A}{2}$ + cot $\frac{B}{2}$ + cot $\frac{C}{2}$ = cot $\frac{A}{2}$ cot $\frac{B}{2}$ cot $\frac{C}{2}$

(iii) tan2A + tan2B + tan2C = tan2A tan2B tan2C

(iv) tan $\frac{A}{2}$ tan $\frac{C}{2}$ + tan $\frac{B}{2}$ tan $\frac{C}{2}$ + tan $\frac{A}{2}$ tan $\frac{B}{2}$ = 1

Ex .4 If A+B+C = 0 then prove that

(i) $\cos^2$C + $\cos^2$ B - $\cos^2$ A = 1+ 2cosA sinB sinC

(ii) cosA + cosB + cosC = 4cos $\frac{A}{2}$ cos $\frac{B}{2}$ cos $\frac{C}{2}$ -1

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